Introduction to Inter-universal Teichmüller theory

Fucheng Tan

RIMS, Kyoto University

2018

To my limited experiences, the following seem to be an option for people who wish to get to know IUT without spending too much time on all the details.

- Regard the anabelian results and the general theory of Frobenioids as blackbox.
- Proceed to read Sections 1, 2 of [EtTh], which is the basis of IUT.
- Read [IUT-I] and [IUT-II] (briefly), so as to know the basic definitions.
- Read [IUT-III] carefully. To make sense of the various definitions/constructions in the second half of [IUT-III], one needs all the previous definitions/results.
- The results in [IUT-IV] were in fact discovered first. Section 1 of [IUT-IV] allows one to see the construction in [IUT-III] in a rather concrete way, hence can be read together with [IUT-III], or even before.

S. Mochizuki, The étale theta function and its Frobenioid-theoretic manifestations.

S. Mochizuki, Inter-universal Teichmüller Theory I, II, III, IV.

p-adic numbers

Look at a prime number p.

- Any $x \in \mathbb{Q}$ can be written as $x = p^n \cdot \frac{a}{b}$ $(p \nmid ab)$.
- Set $|x|_p = p^{-n}$ $(|0|_p := 0)$. This defines a metric on $\mathbb{Q}$.
- Product formula:

$$\prod_{c \in \{\text{primes}, \infty\}} |x|_c = 1, \quad \forall x \in \mathbb{Q}^\times.$$

- $\mathbb{Q}_p$:=completion of $(\mathbb{Q}, |\cdot|_p)$.

$$\begin{aligned} \mathbb{Q}_p &= \left\{ x = \sum_{i \gg -\infty}^{\infty} a_i p^i, \quad a_i \in \{0, 1, \cdots, p-1\} \right\} \\ \mathbb{Z}_p &= \left\{ x = \sum_{i=0}^{\infty} a_i p^i, \quad a_i \in \{0, 1, \cdots, p-1\} \right\} \end{aligned}$$

  Alternatively,

$$\begin{aligned} \mathbb{Z}_p &= \varprojlim \mathbb{Z}/p^n = \{(a_n) \mid a_n \in \mathbb{Z}/p^n, a_m \equiv a_n(p^n), \forall m \geq n\} \\ \mathbb{Q}_p &= (p^{\mathbb{N}})^{-1}\mathbb{Z}_p \\ \mathbb{F}_p &= \mathbb{Z}_p/p = \mathbb{Z}/p \end{aligned}$$

- Any $x \in \mathbb{Q}_p$ has the form

$$x = p^r \cdot u, \ \text{with } r \in \mathbb{Q}, u \in \mathbb{Z}_p^\times.$$

- $(p) \subset \mathbb{Z}_p$ has a divided power structure:

$$\frac{p^n}{n!} \in (p).$$

Fields vs Galois groups

- Number field F:= finite ext. of $\mathbb{Q}$. (Fix a separable closure $\overline{F}$.) $G_F := \mathrm{Gal}(\overline{F}/F)$.
- p-adic local field K := finite ext. of $\mathbb{Q}_p$. $G_K := \mathrm{Gal}(\overline{K}/K)$.
- If $K = F_v$ is a non-arch. completion of F, take $G_K \subset G_F$ (up to conjugation!).
- (Neukirch-Uchida) For two number fields F_1, F_2,
  $$\mathrm{Isom}_{\mathrm{field}}(F_1, F_2) \simeq \mathrm{Out}(G_{F_1}, G_{F_2}) := \mathrm{Isom}_{\mathrm{top.\ gp.}}(G_{F_1}, G_{F_2})/\mathrm{Inn}(G_{F_2}).$$
- (Mochizuki) Let K_1, K_2 be two p-adic local fields.
  $$\mathrm{Isom}_{\mathbb{Q}_p}(K_1, K_2) \xrightarrow{\sim} \mathrm{Out}_{\mathsf{Fil}}(G_{K_1}, G_{K_2})$$
  where Fil on G_{K_i} is given by the higher ramification groups.
  (Not used in IUT.)
- (Uchida) For two function fields F_1, F_2 over finite fields,
  $$\mathrm{Isom}_{\mathrm{field}}(F_1, F_2) \simeq \mathrm{Out}(G_{F_1}, G_{F_2}).$$
- The non-isomorphic Frobenii on function fields/curves induce isomorphisms on Galois/étale fundamental groups. $\rightsquigarrow$

  Frobenius-like vs étale like objects.

- **sub-p-adic field** =field $\hookrightarrow$ finitely generated extension of $\mathbb{Q}_p$.
- Let K be a sub-p-adic field and X_K, Y_K two **hyperbolic curves** over K.

(Mochizuki)

$$\mathrm{Isom}_K(X_K, Y_K) \xrightarrow{\sim} \mathrm{Isom}^{\mathrm{outer}}_{G_k}(\Pi_{X_K}, \Pi_{Y_K}).$$

- X=proper smooth connected curve over $\overline{\mathbb{Q}}$.
  - A **Belyi map** is a dominant map of $\overline{\mathbb{Q}}$-schemes $\phi : X \to \mathbb{P}^1_{\overline{\mathbb{Q}}}$ which is unramified over the **tripod** $\mathbb{P}^1\backslash\{0,1,\infty\}$. The preimage $\phi^{-1}(\mathbb{P}^1\backslash\{0,1,\infty\})$ is called an **Belyi open** of X.
  - (Belyi) There exists at least one Belyi open for X.
  - (Mochizuki) Belyi opens of X form a basis of the Zariski topology of X.
- A hyperbolic orbicurve is **of strictly Belyi type** if it is defined over a number field and is isogenous to a hyperbolic curve of genus 0.

(Mochizuki) Let X be a hyperbolic orbicurve over a number field or a p-adic local field which is of strictly Belyi type.
Starting from the étale/tempered fundamental group (as an abstract top. gp.), one can construct in a "group-theoretic" manner the function field and cusps of X.
The two constructions are compatible with respect to the embeddings of the number field in its nonarchmedean completions.

- X = smooth scheme of finite type over a field $K \hookrightarrow \mathbb{C}$.

$$H^i_{\mathrm{dR}}(X/K) \otimes_K \mathbb{C} \xrightarrow{\sim} H^i(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}.$$

Theorem (Tusji, Faltings, Beilinson).
For X a variety over a p-adic field K, $\exists$ isomorphism compatible with Galois action, filtration, Frobenius, and monodromy:

$$H^i(X_{\overline{K},\text{ét}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{st}} \simeq H^i_{\text{log-cris}}(\mathcal{X}_0/\mathcal{O}_{K_0}) \otimes_{K_0} B_{\mathrm{st}}.$$

Theorem (Faltings-Andreatta-Iovita, T.-Tong). Assume K is absolutely unramified over $\mathbb{Q}_p$. Let $\mathcal{X}$ be a proper smooth formal scheme over $\mathcal{O}_K$, and $X = \mathcal{X}_K$. Let $\mathbb{L}$ be a lisse $\mathbb{Z}_p$-sheaf on $X_{\text{ét}}$ and $\mathcal{E}$ a filtered convergent F-isocrystal on $\mathcal{X}_0/\mathcal{O}_K$. If they are **associated**, then $\exists$ natural isomorphism compatible with Galois action, filtration, and Frobenius action:

$$H^i(X_{\overline{K},\text{ét}}, \mathbb{L}) \otimes_{\mathbb{Z}_p} B_{\mathrm{cris}} \simeq H^i_{\mathrm{cris}}(\mathcal{X}_0/\mathcal{O}_K, \mathcal{E}) \otimes_K B_{\mathrm{cris}}.$$

(Faltings, T.-Tong: The analogue for proper smooth morphisms of smooth formal schemes over $\mathcal{O}_K$ holds.)

The local systems in blue and base fields in red provide some intuition into the theme Frobenius-like vs étale like objects.

IUT is in some sense a **global** simulation of the comparison isomorphism.

One starts with a "suitable" pair of **elliptic curve** E_F over a number field F and a prime number ℓ, and studies it via (the fundamental groups of) certain **hyperbolic curves** surrounding **theta function**.
There are two kinds of symmetry associated to a fixed quotient

$$E_F[\ell] \twoheadrightarrow \mathbb{F}_\ell,$$

which give natural labels on **cusps** of certain hyperbolic curves. The additive symmetry is naturally a subquotient of some geometric fundamental group and assures that the conjugacy of local Galois groups on various values of theta function (at these cusps) are synchronized. This set of theta values at each bad place v (modulo torsion) has the form

$$\{\underline{\underline{q}}_v^{j^2}\}_{j=1,\cdots,\frac{\ell-1}{2}}, \quad \underline{\underline{q}}_v = q_v^{\frac{1}{2\ell}}, \quad q_v = q\text{-parameter of } E \text{ at this place.}$$

The multiplicative symmetry is a subquotient of the absolute Galois group of F_{mod} (the field of moduli of E) and assures that the Kummer-theoretic reconstruction of F_{mod} is compatible with the natural labels on the cusps.

These theta values and the number field F_{mod} will determine the Θ-**pilot object** P_Θ. The main construction of IUT is the **multiradial representation** U_Θ of P_Θ, an orbit under the indeterminacies **(Ind1, 2, 3)**, which (roughly) concern the automorphisms of local Galois groups; automorphisms of local unit groups; change of additive structures of local rings.

The multiradial representation U_Θ is in particular **compatible** with the Θ-**link**, which, at a bad place v, may be regarded as an abstract isomorphism

$$\left(\{\underline{\underline{q}}_v^{j^2}\}\right)^{\mathbb{N}} \cdot \mathcal{O}^{\times\mu}_{\overline{F}_v} \simeq \underline{\underline{q}}_v^{\mathbb{N}} \cdot \mathcal{O}^{\times\mu}_{\overline{F}_v}, \qquad (\mathcal{O}^{\times\mu} = \mathcal{O}^\times/\text{torsion}).$$

As a consequence the so-called **holomorphic hull** $\overline{U}_\Theta \supset U_\Theta$ has arithmetic degree bigger than that determined by q-parameters. This leads to a proof of Vojta conjecture.

- For $i \in I$ in a finite index set, $k_i/\mathbb{Q}_p$ a finite extension with ramification index e_i.
- In IUT, $K = F(E_F[\ell])$, $k_i = K_{w_i}$, $i \in I = \{0, \cdots, j\}$, $j \in \{1, \cdots, \frac{\ell-1}{2}\}$:

$$^I\mathcal{I}_{\mathbb{Q}} = \otimes_{i\in I}(\oplus_{\text{some } v|p} K_v)_i \simeq \oplus(\otimes_{w_i|p, i\in I} K_{w_i}), \quad (\overline{U}_{\Theta})_p \subset \prod_j {}^I\mathcal{I}_{\mathbb{Q}}.$$

- $R_i = \mathcal{O}_{k_i}$, $\mathfrak{d}_i$ the valuation of any generator of the different ideal.

$$R_I = \otimes_{\mathbb{Z}_p} R_i, \quad \log(R_I^\times) = \otimes_{\mathbb{Z}_p} \log(R_i^\times), \quad \mathfrak{d}_I = \sum_{i\in I} \mathfrak{d}_i.$$

$$p^{a_i} R_i \subset \log(R_i^\times) \subset p^{-b_i} R_i, \quad a_i = \frac{\lceil e_i/(p-2)\rceil}{e_i}, b_i = \lfloor \frac{\log(pe_i/(p-1))}{\log p} \rfloor - \frac{1}{e_i}.$$

- Common container of domain/codomain of the p-adic log, called **log-shell**:

$$\mathcal{I} = \frac{1}{p}\log(\mathcal{O}_{K_v}^\times).$$

- **Estimate (Ind1,2,3) simultaneously** ($\widetilde{R}_I$ normalization of R_I in the ring of fractions):

For any automorphism

$$\phi : \log(R_I^\times)_{\mathbb{Q}} \xrightarrow{\sim} \log(R_I^\times)_{\mathbb{Q}} \quad \text{with } \log(R_I^\times) \xrightarrow{\sim} \log(R_I^\times),$$

$$\bigcup_{\phi} \phi\left(p^\lambda \cdot \widetilde{R}_I\right) \subset p^{\lfloor \lambda - \mathfrak{d}_I - a_I \rfloor} \log_p(R_I^\times)) \subset p^{\lfloor \lambda - \mathfrak{d}_I - a_I\rfloor - b_I} \widetilde{R}_I, \quad \lambda = v_p(\underbrace{q^{j^2}}_{=v}).$$

- $p^{\lfloor \lambda - \mathfrak{d}_I - a_I\rfloor - b_I}\widetilde{R}_I \rightsquigarrow$ upper bound for the j-component of log-volume $\mu^{\log}((\overline{U}_{\Theta})_p)$.

[IUT3, Cor. 3.12]: The compatibility between the construction of multiradial representation and the Θ-link $\rightsquigarrow$

$$\sum_p \mu^{\log}((\overline{U}_{\Theta})_p) \geq \text{deg. of arith. l.b. determined by } q\text{-parameters of } E_F.$$

k a finite extension of $\mathbb{Q}_p$, $\mathfrak{X}$ a stable log orbicurve over $\mathrm{Spf}\,\mathcal{O}_k$, with special fiber singular and split, and generic fiber X a smooth log (orbi)curve.

- $\{X_i\}_{i \in \mathbb{N}}$ a cofinal system of finite étale (pointed) covers of X.
- X_i^∞ the topological universal cover of X_i^{an}.
- $\mathrm{Gal}(X_i^\infty/X)$ consists of the (compatible) pairs (u, f) with

$$u \in \mathrm{Aut}(X_i^\infty), \quad f \in \mathrm{Gal}(X_i/X).$$

- $\Pi_X^{\mathrm{tp}} := \varprojlim \mathrm{Gal}(X_i^\infty/X)$, each component endowed with discrete topology.
- Finite topological covers of X_i^{an} are algebraizable $\rightsquigarrow$

$$\widehat{\Pi_X^{\mathrm{tp}}} \xrightarrow{\sim} \Pi_X = \pi_1^{\text{ét}}(X).$$

- For an elliptic curve $E/\overline{k}$ with bad reduction (hence the **Tate uniformization**)

$$\pi_1^{\mathrm{tp}}(E) = \varprojlim_N \mathrm{Gal}(\mathbb{G}_m^{\mathrm{an}}/E) \simeq \varprojlim_N (\mathbb{Z} \times \mu_N),$$

transition maps induced by multiplication by N. Short exact sequence

$$1 \to \widehat{\mathbb{Z}}(1) \to \pi_1^{\mathrm{tp}}(E) \to \mathbb{Z} \to 1.$$

- The association to $X_i \in \mathrm{Cov}^{\text{fét}}(X)$ (for every i) of the category of topological covers of X_i^{an}, $\rightsquigarrow$ a category of tempered covers of X

$$\mathcal{B}^{\mathrm{tp}}(X).$$

- For a connected noetherian (generically scheme-like) algebraic stack X, the category of finite étale coverings of X (morphisms over X):

$$\mathcal{B}(X).$$

- The category of finite sets equipped with continuous Π_X-action:

$$\mathcal{B}(\Pi_X) \quad (\simeq \mathcal{B}(X)).$$

- For a topological group Π, the category of countable discrete sets equipped with a continuous Π-action:

$$\mathcal{B}^{\mathrm{tp}}(\Pi).$$

  If Π is *tempered* (i.e. can be written as an inverse limit of surjections of countable discrete topological groups and the topology admits a countable basis), one can reconstruct Π (category-theorectically) from $\mathcal{B}^{\mathrm{tp}}(\Pi)$. For a *temperoid*, i.e. a category $\mathcal{C} \simeq \mathcal{B}^{\mathrm{tp}}(\Pi)$, denote the resulting topological group by

$$\pi_1(\mathcal{C}).$$

- A morphism $\mathcal{C} \to \mathcal{C}'$ of temperoids is an isomorphism class of functors $\mathcal{C}' \to \mathcal{C}$ preserving finite limits and countable colimits. Thus

$$\mathrm{Hom}\left(\mathcal{B}^{\mathrm{tp}}(\Pi), \mathcal{B}^{\mathrm{tp}}(\Pi')\right) \simeq \mathrm{Hom}^{\mathrm{out}}_{\mathrm{cont}}(\Pi, \Pi').$$

- k = finite extension of $\mathbb{Q}_p$. $\mathfrak{X}$ = stable log curve of type $(1,1)$ over Spf $\mathcal{O}_k$, with special fiber is singular and split and generic fiber X a smooth log curve.
- $$1 \to \Delta_X^{\mathrm{tp}} \to \Pi_X^{\mathrm{tp}} \to G_k \to 1.$$
$$1 \to \widehat{\mathbb{Z}}(1) \to \Delta_X^{\mathrm{ell}} = \frac{\Delta_X}{[\Delta_X, \Delta_X]} \to \widehat{\mathbb{Z}} \to 1,$$
$$1 \to \Delta_\Theta(\simeq \widehat{\mathbb{Z}}(1)) \to \Delta_X^{\Theta} = \frac{\Delta_X}{[\Delta_X, [\Delta_X, \Delta_X]]} \to \Delta_X^{\mathrm{ell}} \to 1,$$
where $\Delta_\Theta := \mathrm{Im}(\wedge^2 \Delta_X^{\mathrm{ell}} \hookrightarrow \Delta_X^{\Theta})$.
- Induce
$$1 \to \widehat{\mathbb{Z}}(1) \to (\Delta_X^{\mathrm{tp}})^{\mathrm{ell}} \to \mathbb{Z} \to 1.$$
$$1 \to \Delta_\Theta \to (\Delta_X^{\mathrm{tp}})^{\Theta} \to (\Delta_X^{\mathrm{tp}})^{\mathrm{ell}} \to 1.$$
- The universal cover of the dual graph of the special fiber of $\mathfrak{X} \rightsquigarrow$
$$\Pi_X^{\mathrm{tp}} \twoheadrightarrow \mathbb{Z},$$
thus a formal scheme $\mathfrak{Y}$ over $\mathfrak{X}$ (generic fiber Y) whose special fiber is a chain of copies of $\mathbb{P}^1$, labeled by $\mathbb{Z}$, joined at $0, \infty$.
- $q_X \in \mathcal{O}_k$ the q-parameter of the elliptic curve associated to X. ($Y^{\mathrm{an}} = \mathbb{G}_{m,k} \backslash \{q_X^{\mathbb{Z}}\}$.)
- Form $k_N = k(\zeta_N, q_X^{1/N})$, $N \geq 1$. The restriction to G_{K_N} of any cuspidal decomposition group of $\Pi_Y^{\mathrm{tp}} \rightsquigarrow$ Galois cover
$$Y_N \to Y.$$
$(1 \to (\Delta_Y^{\mathrm{tp}})^{\mathrm{ell}}/N \to \mathrm{Gal}(Y_N/Y) \to \mathrm{Gal}(k_N/k) \to 1.)$
- Normalization $\mathfrak{Y}_N \to \mathfrak{Y}$ of $\mathfrak{Y}$ in Y_N.
$$\mathrm{Pic}(\mathfrak{Y}_N) \simeq \mathbb{Z}^{\mathrm{Gal}(Y/X)} \simeq \mathbb{Z}^{\mathbb{Z}}.$$
The (isomorphism class of) line bundle determined by $1^{\mathrm{Gal}(Y/X)}$ is denoted by $\mathfrak{L}_N$.

- Divisor of cusps of $\mathfrak{Y}$ $\rightsquigarrow$ section $s_1 \in \Gamma(\mathfrak{Y}, \mathfrak{L}_1)$ (well-defined up to $\cdot \mathcal{O}_k^\times$).
- (With suitable base field extension)
  (i) There exists an N-th root s_N of s_1 on some Galois cover of Y_N.
  (ii) $s_N \rightsquigarrow$ a unique action $\Pi_X^{\mathrm{tp}} \curvearrowright \mathfrak{L}_N$ compatible with s_N.
- $\mathfrak{D}_N$ = effective divisor on $\ddot{\mathfrak{Y}}_N := \mathfrak{Y}_{2N}$ supported on the special fiber and is equal to, at the irr. component-j, the zero locus of $q_X^{j^2/2N}$. $\exists \tau_N \in \Gamma(\ddot{\mathfrak{Y}}_N, \ddot{\mathfrak{L}}_N)$, whose zero locus is equal to $\mathfrak{D}_N$. $\rightsquigarrow$ $\Pi_{\ddot{Y}}^{\mathrm{tp}}$-action on $\ddot{\mathfrak{L}}_N$ preserving τ_N.
- Two Galois actions differ by μ_N-multiples. N varies $\rightsquigarrow$

$$\mathcal{O}_{k_2}^\times \cdot \ddot{\eta}^\Theta \subset H^1(\Pi_{\ddot{Y}}^{\mathrm{tp}}, \widehat{\mathbb{Z}}(1)) \simeq H^1(\Pi_{\ddot{Y}}^{\mathrm{tp}}, \Delta_\Theta).$$

- $\mathfrak{U} \subset \mathfrak{Y}$ the open subscheme obtained by removing the node at irr. component-0 of special fiber, $\ddot{\mathfrak{U}} = \mathfrak{U} \times_{\mathfrak{Y}} \ddot{\mathfrak{Y}}$. $\mathfrak{U} \simeq \widehat{\mathbb{G}}_m \rightsquigarrow$ coordinate $\ddot{U} \in \Gamma(\ddot{\mathfrak{U}}, \mathcal{O}_{\ddot{\mathfrak{U}}}^\times)$.
- Meromorphic function on $\ddot{\mathfrak{Y}}$

$$\ddot{\Theta} = \ddot{\Theta}(\ddot{U}) = \sum_{n \in \mathbb{Z}} (-1)^n q_X^{\frac{1}{2}(n^2+n)} \ddot{U}^{2n+1}, \quad \left(\ddot{\Theta}(q_X^{j/2}\ddot{U}) = (-1)^j q_X^{-j^2/2} \ddot{U}^{-2j} \ddot{\Theta}(\ddot{U})\right),$$

  with zeros of **multiplicity one** and exactly the cusps, and divisor of poles $= \mathfrak{D}_1$.
  (Thus, $\mathcal{O}_{k_2}^\times \cdot \ddot{\eta}^\Theta$ coincides with the Kummer classes associated to $\mathcal{O}_{k_2}^\times \cdot \ddot{\Theta}$.)
- From now on, suppose $k = k_2$ and that X is not k-arithmetic. $\rightsquigarrow$
  Constant multiple rigidity: The pullback via section determined by a 4-torsion point gives

$$\pm \ddot{\eta}^{\Theta, \mathbb{Z}} \subset H^1(\Pi_{\ddot{Y}}^{\mathrm{tp}}, \Delta_\Theta)$$

  which corresponds to $\frac{\ddot{\Theta}}{\ddot{\Theta}(\sqrt{-1})}$.

- k char. 0 field, X a smooth log curve of type $(1,1)$ over k, not k-arithmetic.
- The "theta quotient"

$$\Delta_X^\Theta \simeq \begin{pmatrix} 1 & \widehat{\mathbb{Z}}(1) & \widehat{\mathbb{Z}}(1) \simeq \Delta_\Theta \\ & 1 & \widehat{\mathbb{Z}} \\ & & 1 \end{pmatrix}$$

- $\ell \geq 3$ a prime number. The ℓ-th power map yields the quotient exact sequence

$$1 \to \Delta_{\Theta,\ell}(\simeq \mu_\ell) \to \Delta_{X,\ell}^\Theta \to \Delta_{X,\ell}^{\mathrm{ell}} \to 1.$$

- Let x_0 denote the cusp of X. Consider

$$1 \longrightarrow \Delta_{\Theta,\ell} \longrightarrow D_{x_0,\ell} := \mathrm{Im}(D_{x_0} \hookrightarrow \Pi_X^\Theta \twoheadrightarrow \Pi_{X,\ell}^\Theta) \longrightarrow G_k \longrightarrow 1.$$

A quotient $\Delta_{X,\ell}^{\mathrm{ell}} \twoheadrightarrow Q \quad (\simeq \mathbb{Z}/\ell\mathbb{Z})$ whose restriction to $D_{x_0,\ell}$ is trivial $\rightsquigarrow$ Galois cover $\underline{X} \xrightarrow{Q} X$.

$$\Delta_{\underline{X}}^\Theta \simeq \begin{pmatrix} 1 & \widehat{\mathbb{Z}}(1) & \widehat{\mathbb{Z}}(1) \\ & 1 & \ell\widehat{\mathbb{Z}} \\ & & 1 \end{pmatrix}.$$

- ι_X the inversion of X (relative to the origin x_0), $\iota_{\underline{X}}$ the inversion of $\underline{X}$ relative to *some* cusp of $\underline{X}$ lying over x_0, acting by $+1$ on $\Delta_{\Theta,\ell}$ and -1 on $\Delta_{\underline{X},\ell}^{\mathrm{ell}}$:

$$1 \to \Delta_{\Theta,\ell}(\simeq \mu_\ell) \to \Delta_{\underline{X},\ell}^\Theta \to \Delta_{\underline{X},\ell}^{\mathrm{ell}}(\simeq \mathbb{Z}/\ell\mathbb{Z}) \to 1.$$

- $\rightsquigarrow$ splitting $s_\iota : \Delta_{\underline{X},\ell}^{\mathrm{ell}} \to \Delta_{\underline{X},\ell}^\Theta(\subset \Pi_{\underline{X},\ell}^\Theta)$, hence $D_{x_0,\ell} \simeq \Pi_{\underline{X},\ell}^\Theta/\mathrm{Im}(s_\iota)$ over G_k. In particular, any splitting of $D_{x_0,\ell} \twoheadrightarrow G_k$ determines a cover $\underline{\underline{X}} \xrightarrow{\mu_\ell} \underline{X}$,

$$\Delta_{\underline{\underline{X}}}^\Theta \simeq \begin{pmatrix} 1 & \widehat{\mathbb{Z}}(1) & \ell\widehat{\mathbb{Z}}(1) \\ & 1 & \ell\widehat{\mathbb{Z}} \\ & & 1 \end{pmatrix}.$$

ℓ-th roots of étale theta function

- $$\Pi^{\mathrm{tp}}_{\underline{\underline{X}}} \rightsquigarrow \Pi^{\mathrm{tp}}_{C}, \Pi^{\mathrm{tp}}_{X}, \Pi^{\mathrm{tp}}_{Y}, \Pi^{\mathrm{tp}}_{\ddot{Y}}, \Pi^{\mathrm{tp}}_{\underline{X}}, \Pi^{\mathrm{tp}}_{\underline{C}}.$$
$$\Pi^{\mathrm{tp}}_{\underline{X}} \rightsquigarrow \Pi^{\mathrm{tp}}_{C}, \Pi^{\mathrm{tp}}_{X}, \Pi^{\mathrm{tp}}_{Y}, \Pi^{\mathrm{tp}}_{\ddot{Y}}.$$
- **Assumptions**: (1) The quotient $\Delta^{\Theta}_{X,\ell} \twoheadrightarrow Q$ in the definition of $\underline{X}$ is compatible with the natural quotient $\Pi^{\mathrm{tp}}_{X} \twoheadrightarrow \mathrm{Gal}(Y/X) \simeq \mathbb{Z}$. (2) The choice of splitting of $D_{x_0,\ell} \to G_k$ is compatible with the ± 1-structure on the $\widehat{k^{\times}}$-torsor at x_0 determined by $\ddot{\eta}^{\Theta,\mathbb{Z}}$.
- Always keep in mind (over local or global fields):

$$\begin{tikzcd}
\ddot{\underline{\underline{Y}}} \arrow[r, "\mu_\ell"] \arrow[d, "\mu_2"'] & \ddot{Y} \arrow[d, "\mu_2"'] & \\
\underline{\underline{Y}} \arrow[r, "\mu_\ell"] \arrow[d, "\ell\mathbb{Z}"'] & Y \arrow[d, "\ell\mathbb{Z}"'] \arrow[dr, "\mathbb{Z}"] & \\
\underline{\underline{X}} \arrow[r, "\mu_\ell"] & \underline{X} \arrow[r, "Q"] \arrow[d, "\iota_{\underline{X}}"'] & X \arrow[d, "\iota_X"] \\
& \underline{C} \arrow[r, "\deg \ell"] & C
\end{tikzcd}$$

- μ_ℓ-covers $\underline{\underline{X}} \to \underline{X} \leftrightarrow$ "**ℓ-th root of the étale theta function**":
$$\ddot{\underline{\underline{\eta}}}^{\Theta,\ell\mathbb{Z}\times\mu_2} \subset H^1(\Pi^{\mathrm{tp}}_{\ddot{\underline{\underline{Y}}}}, \ell\Delta_\Theta).$$
- **Labels of cusps** (false for $\underline{X}$):
$$\mathrm{Cusp}(*)/\mathrm{Aut}_k(*) \simeq \mathbb{F}_\ell/\{\pm 1\}, \quad * = \underline{\underline{X}}, \underline{C}.$$
In the following, we shall often use $\underline{\underline{X}}$ in the local case and $\underline{C}$ in the global case.

1. F = number field $\ni \sqrt{-1}$.
2. $X_F = E_F \backslash \{o_{E_F}\}$ once-punctured elliptic curve which admits stable reduction at all nonarchimedean places $\mathbb{V}(F)^{\mathrm{non}}$ of F. Suppose $E_F[6]$ is rational over F.
   C_F = stack-theoretic quotient of X_F by its unique F-involution.
   F_{mod} = field of moduli of X_F, with the set of valuations $\mathbb{V}_{\mathrm{mod}}$.
   $\mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}} \subset \mathbb{V}_{\mathrm{mod}}$ a nonempty set of nonarchimedean places of F_{mod} away from 2, where X_F has bad multiplicative reductions at $\mathbb{V}^{\mathrm{bad}} = \mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}} \times_{\mathbb{V}_{\mathrm{mod}}} \mathbb{V}(F)$.
   Assume F/F_{mod} is Galois of degree prime to ℓ (see below).
3. $\ell \geq 5$ a prime number away from $\mathbb{V}^{\mathrm{bad}}_{\mathrm{mod}}$ and the orders of the q-parameters of E_F. Suppose further

$$G_F \to \mathrm{Aut}(E_F[\ell]) \simeq \mathrm{GL}_2(\mathbb{F}_\ell)$$

   has image containing $\mathrm{SL}_2(\mathbb{F}_\ell)$. The kernel determines a number field K.
4. A chosen section

$$\mathbb{V}_{\mathrm{mod}} \xrightarrow{\sim} \underline{\mathbb{V}} \subset \mathbb{V}(K)$$

   such that at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ the cover $\underline{X}_{\underline{v}} \to X_{\underline{v}}$ is compatible with Tate uniformization. Write

$$\Pi_{\underline{v}} := \Pi^{\mathrm{tp}}_{\underline{\underline{X}}_{\underline{v}}}.$$

5. $\underline{\epsilon}$ is a cusp of $\underline{C}_K$ arising from an element of the quotient (in the definition of $\underline{C}_K$)

$$E_K[\ell] \twoheadrightarrow Q$$

   such that at $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$ the corresponding cusp $\underline{\epsilon}_{\underline{v}}$ of $\underline{C}_{\underline{v}}$ arises from the canonical generator "± 1" under Tate uniformization.

- Given a commutative fs (fine and saturated) monoid Φ, an abelian group B and a morphism of monoids $\mathrm{div} : B \to \Phi^{\mathrm{gp}}$, $\rightsquigarrow$ a category

$$\mathcal{C} = \mathcal{C}(\Phi, B, \mathrm{div})$$

with $\mathrm{Ob}(C) = \mathrm{Ob}(\Phi)$ and a morphism $c \to c'$ in $\mathcal{C}$ a pair $(\phi, b) \in \Phi \times B$ such that

$$c + \phi = c' + \mathrm{div}(b).$$

- Consider the presheaf of commutative fs monoids Φ, presheaf of abelian groups $\mathbb{B}$ and a morphism $\mathrm{div} : \mathbb{B} \to \Phi^{\mathrm{gp}}$ over a category $\mathcal{D}$. The resulting fibered category:

$$\mathcal{C} = \mathcal{C}(\Phi, \mathbb{B}, \mathrm{div}) \to \mathcal{D}.$$

An object $\mathrm{Ob}(\mathcal{C})$ consists of a pair (d, ϕ) with $d \in \mathrm{Ob}(\mathcal{D})$ and $\phi \in \Phi(d)$, and a morphism $(d, \phi) \to (d', \phi')$ consists of $\alpha : d \to d'$ and $\phi \to \alpha^* \phi'$.
- $\mathcal{C} \to \mathcal{D}$ above (thus a monoidal structure $\otimes$) $\rightsquigarrow$ **Frobenioid**:

$$\mathcal{F} = \mathcal{F}(\Phi, \mathbb{B}, \mathrm{div})$$

in which the objects are objects of $\mathcal{C}$ and a morphism $f \to f'$ is a pair (n, β) with $n \in \mathbb{Z}_{\geq 1}$ and $\beta : f^{\otimes n} \to f'$ a morphism in $\mathcal{C}$.
- The base functor is denoted by $\mathcal{F} \to \mathcal{D}$, $\quad f \mapsto \underline{f}$.

$$\mathcal{O}^{\triangleright}(f) := \mathrm{End}_{\mathcal{C}_{\underline{f}}}(f) = \Phi(\underline{f}) \times_{\Phi^{\mathrm{gp}}(\underline{f})} \mathbb{B}(\underline{f}), \quad \mathcal{O}^{\times}(f) := \mathrm{Aut}_{\mathcal{C}_{\underline{f}}}(f).$$

- (***p*-adic Frobenioids**) $\Pi \curvearrowright \mathcal{O}^{\triangleright}$ as a Frobenioid associated to

$$(\Phi, \mathbb{B}, \mathrm{div}) = (\mathcal{O}^{\triangleright}/\mathcal{O}^{\times}, (\mathcal{O}^{\triangleright})^{\mathrm{gp}}, \mathrm{ord} : (\mathcal{O}^{\triangleright})^{\mathrm{gp}} \to (\mathcal{O}^{\triangleright}/\mathcal{O}^{\times})^{\mathrm{gp}}).$$

That is, for L/k a finite extension $\in \mathcal{D} = \mathcal{B}(G_k)$, $\Phi(L) = \mathcal{O}_L^{\triangleright}/\mathcal{O}_L^{\times} \simeq \mathbb{Z}_{\geq 0}$, $\mathbb{B}(L) = L^{\times}$, $\mathrm{div} = \mathrm{ord} : L^{\times} \to \mathbb{Z}$. Of course, in this case $\mathcal{O}^{\triangleright}(L) = \mathcal{O}_L^{\triangleright}$, $\mathcal{O}^{\times}(L) = \mathcal{O}_L^{\times}$.

- Let F be a number field and $\widetilde{F}/F$ a Galois extension with Galois group G. Let L be a finite extension of F inside $\widetilde{F}$, regarded as an object of $\mathcal{D} = \mathcal{B}(G)$. Define

$$\Phi(L) = \oplus_{v \in \mathbb{V}(L)} \mathcal{O}^{\triangleright}_{L_v}/\mathcal{O}^{\times}_{L_v}, \quad B(L) = L^{\times}, \quad \mathrm{div} : L \to \Phi(L)^{\mathrm{gp}},$$

where at an archimedean place $\mathcal{O}^{\triangleright}_{L_v}/\mathcal{O}^{\times}_{L_v} := \mathbb{R}_{\geq 0}$. $\rightsquigarrow$ Frobenioid

$$\mathcal{F} = \mathcal{F}(\overline{F}/F) \longrightarrow \mathcal{D} = \mathcal{B}(G).$$

We may then regard an object in $\mathcal{F}_L$ as an arithmetic line bundle on $\operatorname{Spec} \mathcal{O}_L$.
- X proper normal variety over a field k, $\widetilde{K}/K$ a Galois extension of the function field $K = K_X$ of X with Galois group G, L/K a finite extension. Let $\mathbb{D}_K$ be a set of $\mathbb{Q}$-Cartier prime divisors on X and $\mathbb{D}_L$ the set of prime divisors on $X[L]$ (the normalization of X in L) mapping into $\mathbb{D}_K$, which are all assumed $\mathbb{Q}$-Cartier. Define a Frobenioid

$$\mathcal{F} = \mathcal{F}(X, \widetilde{K}, \mathbb{D}_K) \to \mathcal{D} = \mathcal{B}(G) :$$

  - $\Phi(L)$ = monoid of effective Cartier divisors on $X[L]$ with support in $\mathbb{D}_L$;
  - $\mathbb{B}(L)$ = group of rational functions b on $X[L]$ so that each prime divisor where b has a zero/pole belongs to $\mathbb{D}_L$;
  - $\mathrm{div} : \mathbb{B}(L) \to \Phi(L)^{\mathrm{gp}}$ the natural homomorphism.

k a finite extension of $\mathbb{Q}_p$, $\mathfrak{X}$ a stable log orbicurve over $\mathrm{Spf}\,\mathcal{O}_k$ with special fiber singular and split and generic fiber X is a smooth log (orbi)curve.

- Consideration of tempered covers of $X \rightsquigarrow$ presheaf of momnoids/groups on $\mathcal{D}_X = \mathcal{B}^{\mathrm{tp}}(X)$ and natural homomorphism (image denoted by Φ^{birat}):

  $$\Phi_0, \quad \mathbb{B}_0, \quad \mathrm{div} : \mathbb{B}_0 \to \Phi_0^{\mathrm{gp}}, \quad \rightsquigarrow$$

  tempered Frobenioid (fibered category)

  $$\mathcal{F}_0 \to \mathcal{D}_X.$$

- Suppose X is of type $(1,1)$. Considering the full subcategory of $\mathcal{D}_X$ of tempered coverings unramified over the cusps and the generic points of irr. components of special fiber and modifying the monoid Φ_0 above accordingly $\rightsquigarrow$ tempered Frobenioid

  $$\underline{\underline{\mathcal{F}}} \to \mathcal{D} = \mathcal{B}^{\mathrm{tp}}(\underline{\underline{X}}).$$

- Base categories

$$\mathcal{D}_{\underline{v}} := \mathcal{B}^{\mathrm{tp}}(\Pi_{\underline{\underline{X}}_{\underline{v}}}), \quad \mathcal{D}^{\vdash}_{\underline{v}} := \mathcal{B}(G_{K_{\underline{v}}}) \subset \mathcal{D}_{\underline{v}}.$$

Natural functor left adjoint to the inclusion above

$$\mathcal{D}_{\underline{v}} \to \mathcal{D}^{\vdash}_{\underline{v}}.$$

- $\underline{\underline{\mathcal{F}}}_{\underline{v}} \rightsquigarrow$ (up to $\mu_{2\ell} \times \ell\mathbb{Z}$-indeterminacy) the *reciprocal* of ℓ-th root of theta function

$$\underline{\underline{\Theta}}_{\underline{v}} \in K^{\times}_{\underline{\underline{\ddot{Y}}}_{\underline{v}}}.$$

- Constant meromorphic functions$\underline{\underline{\mathcal{F}}}_{\underline{v}} \rightsquigarrow$ the base-field-theoretic hull given by :

$$\mathcal{C}_{\underline{v}} \subset \underline{\underline{\mathcal{F}}}_{\underline{v}}.$$

- The theta value

$$\underline{\underline{\Theta}}_{\underline{v}}(\sqrt{-q_{\underline{v}}}) = q_{\underline{v}}^{\frac{1}{2\ell}} =: \underline{\underline{q}}_{\underline{v}}, \quad \text{up to } \mu_{2\ell}\text{-multiples},$$

determines a constant section (over $\mathcal{D}_{\underline{v}}$) of the divisor monoid $\Phi_{\mathcal{C}_{\underline{v}}}$ of $\mathcal{C}_{\underline{v}}$, denoted by

$$\mathbb{N} \cdot \log_{\Phi}(\underline{\underline{q}}_{\underline{v}}) \subset \Phi_{\mathcal{C}_{\underline{v}}},$$

thus a $p_{\underline{v}}$-adic Frobenioid and a $\mu_{2\ell}$-orbit of splittings, i.e. a **split Frobenioids**:

$$\mathcal{F}^{\vdash}_{\underline{v}} := (\mathcal{C}^{\vdash}_{\underline{v}}, \tau^{\vdash}_{\underline{v}}).$$

- Holomorphic/mono-analytic $\mathcal{F}$-**prime-strip**

$$\mathfrak{F} = \{\mathcal{F}_{\underline{v}} = {}^{\dagger}\mathcal{C}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}, \quad \mathfrak{F}^{\vdash} = \{\mathcal{F}^{\vdash}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}.$$

- **Realification** of the Frobenioid associated to (F_{mod}, the trivial extension):

$$\mathcal{C}^{\Vdash}_{\mathrm{mod}} :$$

at each $v \in \mathbb{V}_{\mathrm{mod}}$,

$$\Phi_{\mathcal{C}^{\Vdash}_{\mathrm{mod},v}} \simeq \mathrm{ord}(\mathcal{O}^{\rhd}_{F_{\mathrm{mod},v}})^{\mathrm{pf}} \otimes \mathbb{R}_{\geq 0} \quad (\simeq \mathbb{R}_{\geq 0}).$$

The restriction functor $\mathcal{C}^{\Vdash}_{\mathrm{mod}} \to (\mathcal{C}^{\vdash}_{\underline{v}})^{\mathrm{rlf}}$ induces (for $\underline{v}|v$)

$$\rho_{\underline{v}} : \Phi_{\mathcal{C}^{\Vdash}_{\mathrm{mod},v}} \xrightarrow{\sim} \Phi^{\mathrm{rlf}}_{\mathcal{C}^{\vdash}_{\underline{v}}}, \quad \log^{\vdash}_{\mathrm{mod}}(p_v) \mapsto \frac{\log_{\Phi}(p_{\underline{v}})}{[K_{\underline{v}} : F_{\mathrm{mod},v}]}.$$

- **Global realified mono-analytic Frobenioid-prime-strip**

$$\mathfrak{F}^{\Vdash} = \mathfrak{F}^{\Vdash}_{\mathrm{mod}} := \left(\mathcal{C}^{\Vdash}_{\mathrm{mod}}, \mathrm{Prime}(\mathcal{C}^{\Vdash}_{\mathrm{mod}}) \simeq \underline{\mathbb{V}}, \mathfrak{F}^{\vdash} = \{\mathcal{F}^{\vdash}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}, \{\rho_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}\right).$$

$\rightsquigarrow$ a well-defined degree map on $\mathfrak{F}^{\Vdash}$, invariant under automorphisms.

-

$$\mathcal{HT}^{\Theta} := \left(\{\underline{\underline{\mathcal{F}}}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}, \quad \mathfrak{F}^{\Vdash}_{\mathrm{mod}}\right).$$

- $$\mathcal{D}^{\circledcirc} := \mathcal{B}(\underline{C}_K).$$
  $$\overline{F}(\mathcal{D}^{\circledcirc}) := \overline{F}^{\times}(\mathcal{D}^{\circledcirc}) \cup \{0\} \simeq \overline{F}.$$
- The (unique) model $C_{F_{\mathrm{mod}}}$ over F_{mod} of $C_F \rightsquigarrow$
  $$(\pi_1(\mathcal{D}^{\circledcirc}) \subset) \quad \pi_1(\mathcal{D}^{\circledast}) \simeq \Pi_{C_{F_{\mathrm{mod}}}}, \quad \mathcal{D}^{\circledast} := \mathcal{B}(\pi_1(\mathcal{D}^{\circledast})).$$
  $$F^{\times}_{\mathrm{mod}}(\mathcal{D}^{\circledcirc}) \simeq F^{\times}_{\mathrm{mod}}, \quad F_{\mathrm{mod}}(\mathcal{D}^{\circledcirc}) \simeq F_{\mathrm{mod}}.$$
- (κ-**coric functions**) Let $L = F_{\mathrm{mod}}$. (Note $|C_L| \simeq \mathbb{A}^1_L$.) For the curve determined by some finite extension of the function field K_C, a closed point is called critical if it maps to the 2-torsion points of E_F. A critical point not mapping to the cusp of C_L is called strictly critical.
  (i) A rational function $f \in K_C$ is κ-coric if
  - if $f \notin L$, then it has exactly 1 pole and ≥ 2 (distinct) zeros,
  - the divisor of zeros and poles of f is defined over a number field and avoids the critical points, and
  - f restricts to roots of unity at strictly critical points of $|C_L|^{\mathrm{cpt}}$.

  (ii) ${}_{\infty}\kappa$-coric if f^n is κ-coric for some $n \in \mathbb{Z}_{>0}$.
  (iii) ${}_{\infty}\kappa\times$-coric if $c \cdot f$ is ${}_{\infty}\kappa$-coric for some $c \in \overline{L}^{\times}$.
- $\pi_1(\mathcal{D}^{\circledcirc}) \rightsquigarrow$ a profinite group
  $$(G_{K_{C_{F_{\mathrm{mod}}}}}) \simeq \quad \pi_1^{\mathrm{rat}}(\mathcal{D}^{\circledast})(\twoheadrightarrow \pi_1(\mathcal{D}^{\circledast})),$$
  pseudo-monoids of κ-, ${}_{\infty}\kappa$- and ${}_{\infty}\kappa\times$-rational functions:
  $$\mathbb{M}^{\circledast}_{\kappa}(\mathcal{D}^{\circledcirc}), \quad \mathbb{M}^{\circledast}_{\infty\kappa}(\mathcal{D}^{\circledcirc}), \quad \mathbb{M}^{\circledast}_{\infty\kappa\times}(\mathcal{D}^{\circledcirc}).$$

- Now the assignment

$$\mathrm{Ob}(\mathcal{D}^{\circledast}) \ni H \mapsto \text{monoid of arithmetic divisors on } \overline{F}(\mathcal{D}^{\circledcirc})^{H}$$

$\rightsquigarrow$ Frobenioid over $\mathcal{D}^{\circledast}$:

$$\mathcal{F}^{\circledast}(\mathcal{D}^{\circledcirc}).$$

- For ${}^{\dagger}\mathcal{F}^{\circledast} \simeq \mathcal{F}^{\circledast}(\mathcal{D}^{\circledcirc})$ a category with base $\mathcal{D}^{\circledast}$, define

$${}^{\dagger}\mathcal{F}^{\circledast}_{\mathrm{mod}} := {}^{\dagger}\mathcal{F}^{\circledast}|_{\text{terminal objects of } \mathcal{D}^{\circledast}}.$$

Note

$${}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{mod}} \simeq \mathcal{C}^{\Vdash}_{\mathrm{mod}}.$$

- Look at an **isomorph**

$$\pi_1^{\mathrm{rat}}({}^{\dagger}\mathcal{D}^{\circledast}) \curvearrowright {}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa\times}.$$

- $$\mu^{\text{ét}}(\pi_1({}^{\dagger}\mathcal{D}^{\circledcirc})) := \mathrm{Hom}(H^2(\Delta_Z, \widehat{\mathbb{Z}}), \widehat{\mathbb{Z}})$$

for Z the canonical compactification of some hyperbolic curve of genus ≥ 2 finite étale over $\underline{X}_K$ appearing in the Belyi cuspidalization of $\underline{X}_K$.

- $$\mu^{\mathrm{Fr}}((-)) = \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, (-)^{\times}).$$

- The consideration of divisors of zeros and poles of κ-coric functions and the fact

$$\mathbb{Q}_{>0} \cap \widehat{\mathbb{Z}}^{\times} = \{1\},$$

$\rightsquigarrow$

$\exists!$ isomorphism of cyclotomes

$$\mu^{\text{ét}}(\pi_1({}^{\dagger}\mathcal{D}^{\circledcirc})) \xrightarrow{\sim} \mu^{\text{Fr}}({}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa\times})$$

making the following diagram commute:

$$\begin{tikzcd}
\mathbb{M}^{\circledast}_{\infty\kappa\times}({}^{\dagger}\mathcal{D}^{\circledcirc}) \arrow[r, hook] \arrow[d, "\simeq"'] & \varinjlim_H H^1(H, \mu^{\text{ét}}(\pi_1({}^{\dagger}\mathcal{D}^{\circledcirc}))) \arrow[d, "\simeq"'] \\
{}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa\times} \arrow[r, hook] & \varinjlim_H H^1(H, \mu^{\text{Fr}}({}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa\times}))
\end{tikzcd}$$

- We conclude that ${}^{\dagger}\mathcal{F}^{\circledast}$ carries natural structures of ${}_{\infty}\kappa\times$, ${}_{\infty}\kappa$-, κ-rational functions

$$\pi_1^{\text{rat}}({}^{\dagger}\mathcal{D}^{\circledast}) \curvearrowright {}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa\times}, \quad {}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa}, \quad {}^{\dagger}\mathbb{M}^{\circledast}_{\kappa} := ({}^{\dagger}\mathbb{M}^{\circledast}_{\infty\kappa})^{\pi_1^{\text{rat}}({}^{\dagger}\mathcal{D}^{\circledast})}.$$

(Under the additional requirement of *compatiblility with local integral submonoids*, one has the analogue for $\overline{F}$.)

- In the local case, the p-adic Frobenioid $\mathcal{F}_{\underline{v}}$ carries the analogous structures (via the curve $C_v, v \in \mathbb{V}_{\text{mod}}$).

- For each $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{non}}$, $\mathcal{D}_{\underline{v}} \simeq \mathcal{B}^{\mathrm{tp}}(\underline{\underline{X}}_{\underline{v}})$. (At each $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{arc}}$...)
  Prime-strips:
$$\mathfrak{D} := \{\mathcal{D}_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}},$$
$$\mathfrak{D}^{\vdash} := \{\mathcal{D}^{\vdash}_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}.$$
- A **label class of cusps** of $\mathcal{D}_{\underline{v}}$ is the set of cusps of $\mathcal{D}_{\underline{v}}$ over a (single) cusp of $\underline{\mathcal{D}}_{\underline{v}} \simeq \mathcal{B}^{\mathrm{tp}}(\Pi_{\underline{C}_{\underline{v}}})$ arising from a *nonzero* element of the quotient $Q \simeq \mathbb{F}_\ell$.
- At a bad place $\underline{v}$, the action of $\mathrm{Aut}_{K_{\underline{v}}}(\underline{\underline{X}}_{\underline{v}})$ on the set of cusps of $\underline{\underline{X}}_{\underline{v}}$ factors through its quotient $\{\pm 1\}$.
- At each $\underline{v} \in \underline{\mathbb{V}}$, the action of $\mathbb{F}_\ell^\times \curvearrowright Q$ equips the set $\mathrm{LabCusp}(\mathcal{D}_{\underline{v}})$ with an $\mathbb{F}_\ell^{\circledast}$-torsor structure, where
$$\mathbb{F}_\ell^{\circledast} := (\mathbb{F}_\ell \backslash \{0\})/\{\pm 1\}, \quad |\mathbb{F}_\ell^{\circledast}| = \ell^{\circledast} := \frac{\ell - 1}{2}.$$
-
$$\mathrm{LabCusp}({}^\dagger\mathcal{D}_{\underline{v}}) \xrightarrow{\sim} \mathrm{LabCusp}({}^\dagger\mathcal{D}_{\underline{w}}),$$
$$\rightsquigarrow \quad \mathrm{LabCusp}({}^\dagger\mathfrak{D}).$$

- The existence of the model $C_{F_{\mathrm{mod}}} \rightsquigarrow$

$$\mathsf{Gal}(K/F_{\mathrm{mod}}) \longrightarrow \mathsf{GL}_2(\mathbb{F}_\ell)/\{\pm 1\},$$

$$\begin{aligned} \mathsf{Aut}(\underline{C}_K) &\xrightarrow{\sim} \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\} /\{\pm 1\} \bigcap \mathrm{Im}(\mathsf{Gal}(K/F_{\mathrm{mod}})), \\ \mathsf{Aut}_{\underline{\epsilon}}(\underline{C}_K) &\xrightarrow{\sim} \left\{\begin{pmatrix} * & * \\ 0 & \pm 1 \end{pmatrix}\right\} /\{\pm 1\} \bigcap \mathrm{Im}(\mathsf{Gal}(K/F_{\mathrm{mod}})), \end{aligned}$$

$$\mathsf{Aut}(\underline{C}_K)/\,\mathsf{Aut}_{\underline{\epsilon}}(\underline{C}_K) \simeq \mathsf{Aut}(\mathcal{D}^{\circledcirc})/\,\mathsf{Aut}_{\underline{\epsilon}}(\mathcal{D}^{\circledcirc}) \xrightarrow{\sim} \mathbb{F}_\ell^{\divideontimes}.$$

- For $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, the consideration of the covers $\underline{\underline{X}}_{\underline{v}} \to \underline{C}_{\underline{v}} \to \underline{C}_K$ yields morphisms

$$\phi^{\mathrm{NF}}_{\bullet,\underline{v}} : \mathcal{D}_{\underline{v}} \to \mathcal{D}^{\circledcirc}.$$

- Set:

$$\phi^{\mathrm{NF}}_{\underline{v}} = \{\beta \circ \phi^{\mathrm{NF}}_{\bullet,\underline{v}} \circ \alpha\}_{\alpha \in \mathsf{Aut}(\mathcal{D}_{\underline{v}}),\quad \beta \in \mathsf{Aut}_{\underline{\epsilon}}(\mathcal{D}^{\circledcirc})},$$

$$\phi^{\mathrm{NF}}_j : \mathfrak{D}_j = \{(\mathcal{D}_{\underline{v}})_j\}_{\underline{v} \in \underline{\mathbb{V}}} \to \mathcal{D}^{\circledcirc},$$

$$\mathbb{F}_\ell^{\divideontimes}\text{-equivariant}: \quad \phi^{\mathrm{NF}}_{\divideontimes} = \{\phi^{\mathrm{NF}}_j\} : \mathfrak{D}_{\divideontimes} = \{\mathfrak{D}_j\}_{j \in \mathbb{F}_\ell^{\divideontimes}} \to \mathcal{D}^{\circledcirc}.$$

- Let $\mu_- \in \underline{X}_{\underline{v}}(K_{\underline{v}})$ be the unique 2-torsion point whose closure intersects irr. component-0 (of the special fiber of (any) given stable model) of $\underline{X}_{\underline{v}}$. Define the **evaluation points** of $\underline{X}_{\underline{v}}(K_{\underline{v}})$ as μ_--translations of the cusps.
- This way, the value of the function $\underline{\underline{\Theta}}_{\underline{v}}$ at an evaluation point of $\underline{\underline{\ddot{Y}}}_{\underline{v}}$ with label $j \in |\mathbb{F}_\ell|$ lies in the $\mu_{2\ell}$-orbit of $\underline{\underline{q}}_{\underline{v}}^{j^2}$.
- For any $j \in \mathbb{F}_\ell^{\circledast}$ and $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, we have the poly-morphism

$$\phi^{\Theta}_{\underline{v}_j} : (\mathcal{D}_{\underline{v}})_j \xrightarrow{\sim} \mathcal{B}^{\mathrm{tp}}(\Pi_{\underline{v}}) \to \mathcal{B}(G_{\underline{v}}) \to \mathcal{B}^{\mathrm{tp}}(\Pi_{\underline{v}}) \xrightarrow{\sim} \mathcal{D}_{>,\underline{v}}$$

  with the middle arrows induced by the evaluation section labeled by j and the natural surjection $\Pi_{\underline{v}} \twoheadrightarrow G_{\underline{v}}$.
- To encode the symmetry (involving all possible trivializations of the $\mathbb{F}_\ell^{\circledast}$-torsor $\mathrm{LabCusp}(\mathcal{D}_{\underline{v}})$):

$$\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}} = (\mathcal{D}^{\circledcirc} \xleftarrow{\phi_{\circledast}^{\mathrm{NF}}} \mathfrak{D}_J \xrightarrow{\phi_{\circledast}^{\Theta}} \mathfrak{D}_>).$$

- $\exists [\underline{\epsilon}] \in \mathrm{LabCusp}(\mathcal{D}^{\circledcirc})$ s.t. under the canonical bijection $\mathrm{LabCusp}(\mathfrak{D}_>) \xrightarrow{\sim} \mathbb{F}_\ell^{\circledast}$,

$$\phi_j^{(-)}([\underline{\epsilon}]) = \phi_1^{(-)}(j \cdot [\underline{\epsilon}]) \mapsto j.$$

  $\rightsquigarrow$**Synchronized labelling:**

$$\mathrm{LabCusp}(\mathcal{D}^{\circledcirc}) \xrightarrow{\phi_j^{\mathrm{NF}}} \mathrm{LabCusp}(\mathfrak{D}_j) \xrightarrow{\phi_j^{\Theta}} \mathrm{LabCusp}(\mathfrak{D}_>) \overset{\text{can.}}{\xrightarrow{\sim}} \mathbb{F}_\ell^{\circledast}$$

  $\phi_{\circledast}^{\Theta} : \mathfrak{D}_J \to \mathfrak{D}_>$ determines (uniquely) $\psi_{\circledast}^{\Theta} : \mathfrak{F}_J \longrightarrow \mathfrak{F}_>$.
- (Assume the prime-strip $\mathfrak{F}_>$ associated to $\mathcal{HT}^{\Theta}$ (via the $\underline{\underline{\mathcal{F}}}_{\underline{v}}$) has base $\mathfrak{D}_>$.)

$$\mathcal{HT}^{\Theta\mathrm{NF}} = (\mathcal{F}^{\circledast} \dashleftarrow \mathcal{F}^{\circledcirc} \xleftarrow{\psi_{\circledast}^{\mathrm{NF}}} \mathfrak{F}_J \xrightarrow{\psi_{\circledast}^{\Theta}} \mathfrak{F}_> \dashrightarrow \mathcal{HT}^{\Theta}).$$

$\pm$-label classes of cusps of geometric nature

- An $\mathbb{F}_\ell^{\pm}$-torsor is a set $T \simeq \mathbb{F}_\ell$, where $\mathbb{F}_\ell^{\rtimes\pm} := \mathbb{F}_\ell \rtimes \{\pm 1\}$ acts on $\mathbb{F}_\ell$ by $z \mapsto \pm z + \lambda$ for $\lambda \in \mathbb{F}_\ell$.
- A $\pm$-**label class of cusps** of $\mathcal{D}_{\underline{v}}$ is the set of cusps of $\mathcal{D}_{\underline{v}}$ lying over a single cusp of (the temperoid of) $\underline{X}_{\underline{v}}$.
- There are two canonical elements in $\mathrm{LabCusp}^{\pm}(\mathcal{D}_{\underline{v}})$, corresponding to the zero cusp and the canonical generator of Q. $\rightsquigarrow$

$$\left\{\mathrm{LabCusp}^{\pm}(\mathcal{D}_{\underline{v}}) \backslash \{0\text{-cusp}\}\right\} / \{\pm 1\} \xrightarrow{\sim} \mathrm{LabCusp}(\mathcal{D}_{\underline{v}}) \quad (\xrightarrow{\sim} \mathbb{F}_\ell^{*}).$$

- Now the global situation.

$$\mathcal{D}^{\circledcirc\pm} := \mathcal{B}(\underline{X}_K).$$

Homomorphism (adapted to the quotient Q):

$$\mathrm{Aut}(\mathcal{D}^{\circledcirc\pm}) \xrightarrow{\sim} \mathrm{Aut}(\underline{X}_K) \to \mathrm{GL}_2(\mathbb{F}_\ell) \to \mathrm{GL}_2(\mathbb{F}_\ell)/\{\pm 1\}.$$

$$\mathrm{Aut}_{\pm}(\mathcal{D}^{\circledcirc\pm}) := \ker\left(\mathrm{Aut}(\mathcal{D}^{\circledcirc\pm}) \twoheadrightarrow \mathbb{F}_\ell^{*}\right),$$

$$(\Delta_{C_K}/\Delta_{\underline{X}_K} \simeq) \quad \mathrm{Aut}_{K}(\underline{X}_K) \xrightarrow{\sim} \mathrm{Aut}_{\pm}(\mathcal{D}^{\circledcirc\pm})/\,\mathrm{Aut}_{\mathrm{csp}}(\mathcal{D}^{\circledcirc\pm}) \overset{\epsilon}{\xrightarrow{\sim}} \mathbb{F}_\ell^{\rtimes\pm}.$$

- $\rightsquigarrow$ isomorphism of $\mathbb{F}_\ell^{\pm}$-torsors

$$\mathrm{LabCusp}^{\pm}(\mathcal{D}^{\circledcirc\pm}) \simeq \mathbb{F}_\ell.$$

- The non-interference with local Galois groups of the following action

$$\Delta_{C_K}/\Delta_{\underline{X}_K} \curvearrowright \mathrm{LabCusp}^{\pm}(\mathcal{D}^{\circledcirc\pm}) \simeq \mathrm{LabCusp}^{\pm}({}^{\dagger}\mathcal{D}_{\underline{v}_t}), \forall t \in \mathbb{F}_\ell$$

$\rightsquigarrow$ the **conjugate synchronization** for theta values on various cusps.

We use $^{\dagger}(-)$ to indicate the group/category-theoretic nature of the constructions.

- Recall

$$\begin{aligned}
{}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta\mathrm{NF}} &= ({}^{\dagger}\mathcal{D}^{\circledcirc} \xleftarrow{{}^{\dagger}\phi_{\circledast}^{\mathrm{NF}}} {}^{\dagger}\mathfrak{D}_{J} \xrightarrow{{}^{\dagger}\phi_{\circledast}^{\Theta}} {}^{\dagger}\mathfrak{D}_{>}), \\
{}^{\dagger}\mathcal{HT}^{\Theta\mathrm{NF}} &= ({}^{\dagger}\mathcal{F}^{\circledast} \dashleftarrow {}^{\dagger}\mathcal{F}^{\circledcirc} \xleftarrow{{}^{\dagger}\psi_{\circledast}^{\mathrm{NF}}} {}^{\dagger}\mathfrak{F}_{J} \xrightarrow{{}^{\dagger}\psi_{\circledast}^{\Theta}} {}^{\dagger}\mathfrak{F}_{>} \dashrightarrow {}^{\dagger}\mathcal{HT}^{\Theta}),
\end{aligned}$$

- Similarly, we encode the additive symmetry

$$\begin{aligned}
{}^{\dagger}\mathcal{HT}^{\mathcal{D}\text{-}\Theta^{\pm}\mathrm{ell}} &= ({}^{\dagger}\mathcal{D}^{\circledcirc\pm} \xleftarrow{{}^{\dagger}\phi_{\pm}^{\Theta^{\mathrm{ell}}}} {}^{\dagger}\mathfrak{D}_{T} \xrightarrow{{}^{\dagger}\phi_{\pm}^{\Theta^{\pm}}} {}^{\dagger}\mathfrak{D}_{\succ}), \\
{}^{\dagger}\mathcal{HT}^{\Theta^{\pm}\mathrm{ell}} &= ({}^{\dagger}\mathcal{D}^{\circledcirc\pm} \xleftarrow{{}^{\dagger}\psi_{\pm}^{\Theta^{\mathrm{ell}}}} {}^{\dagger}\mathfrak{F}_{T} \xrightarrow{{}^{\dagger}\psi_{\pm}^{\Theta^{\pm}}} {}^{\dagger}\mathfrak{F}_{\succ}).
\end{aligned}$$

- $T = \mathbb{F}_{\ell}$. Write $|T| = T/\{\pm 1\}$, $\quad T^{\circledast} := |T| \backslash \{0\}$. Glue

$$\begin{aligned}
({}^{\dagger}\phi_{\pm}^{\Theta^{\pm}} : {}^{\dagger}\mathfrak{D}_{T} \to {}^{\dagger}\mathfrak{D}_{\succ}) &\to ({}^{\dagger}\phi_{\circledast}^{\Theta} : {}^{\dagger}\mathfrak{D}_{T^{\circledast}} \to {}^{\dagger}\mathfrak{D}_{>}) \\
{}^{\dagger}\mathfrak{D}_{T}|_{T\backslash\{0\}} &\mapsto {}^{\dagger}\mathfrak{D}_{T^{\circledast}}, \\
{}^{\dagger}\mathfrak{D}_{0}, {}^{\dagger}\mathfrak{D}_{\succ} &\mapsto {}^{\dagger}\mathfrak{D}_{>}.
\end{aligned}$$

$\rightsquigarrow$ the Frobenius-like/étale like Hodge-theaters

$$ {}^{\dagger}\mathcal{HT}, \quad {}^{\dagger}\mathcal{HT}^{\mathcal{D}}. $$

- Such a gluing is unique.

- Let N be a positive integer.
- Take (a cocycle of) the reduction mod N of one class in $\underline{\underline{\ddot{\eta}}}^{\Theta,\ell\mathbb{Z}\times\mu_2} \subset H^1(\Pi^{\mathrm{tp}}_{\underline{\underline{\ddot{Y}}}}, \ell\Delta_\Theta)$ and subtract it from the tautological section

$$s^{\mathrm{tau}}_{\underline{\underline{\ddot{Y}}}} : \Pi^{\mathrm{tp}}_{\underline{\underline{\ddot{Y}}}} \to \Pi^{\mathrm{tp}}_{\underline{\underline{\ddot{Y}}}}[\mu_N] \hookrightarrow \Pi^{\mathrm{tp}}_{\underline{\underline{Y}}}[\mu_N] = \Pi^{\mathrm{tp}}_{\underline{\underline{Y}}} \times_{G_k} (\mu_N \rtimes G_k)$$

This yields a homomorphism

$$s^{\Theta}_{\underline{\underline{\ddot{Y}}}} : \Pi^{\mathrm{tp}}_{\underline{\underline{\ddot{Y}}}} \to \Pi^{\mathrm{tp}}_{\underline{\underline{Y}}}[\mu_N].$$

- Modifying a cocycle above by coboundaries $\leftrightarrow$ replacing $s^{\Theta}_{\underline{\underline{\ddot{Y}}}}$ by its μ_N-conjugates.
- A mod N **Mono-theta environment**

$$\mathbb{M}^{\Theta} = \{\Pi, \mathcal{D}_\Pi, s_\Pi\} :$$

(i) a topological group $\Pi \simeq \Pi^{\mathrm{tp}}_{\underline{\underline{Y}}}[\mu_N]$;
(ii) a subgroup $\mathcal{D}_\Pi \subset \mathrm{Out}(\Pi)$ such that $\mathcal{D}_\Pi \simeq \mathcal{D}_{\underline{\underline{Y}}} \subset \mathrm{Out}(\Pi^{\mathrm{tp}}_{\underline{\underline{Y}}}[\mu_N])$, generated by the images of $K^\times, \mathrm{Gal}(\underline{\underline{Y}}/\underline{\underline{X}})$ via the natural outer actions;
(iii) a collection of subgroups $s^{\Theta}_{\Pi} \subset \Pi$, isomorphic to the μ_N-conjugacy class of $s^{\Theta}_{\underline{\underline{\ddot{Y}}}}(\Pi^{\mathrm{tp}}_{\underline{\underline{\ddot{Y}}}})$.

- Can recover

$$\Pi^{\mathrm{tp}}_{\underline{\underline{X}}} = \mathrm{Aut}(\Pi^{\mathrm{tp}}_{\underline{\underline{Y}}}[\mu_N]) \times_{\mathrm{Out}(\Pi^{\mathrm{tp}}_{\underline{\underline{Y}}}[\mu_N])} \mathcal{D}_{\underline{\underline{Y}}}.$$

The data (ii, iii) in $\mathbb{M}^{\Theta}$ has the use of rigidifying data (i), which give rise to a unique isomorphism of cyclotomes identifying the Frobenius-like and étale-like theta classs.

- (**Cyclotomic rigidity**) $\mathbb{M}^{\Theta} \rightsquigarrow$ isomorphs of splittings $s^{\Theta}_{\underline{\underline{\ddot{Y}}}}, s^{\mathrm{tau}}_{\underline{\underline{\ddot{Y}}}}$. Their difference $(g \mapsto s^{\Theta}(g)/s^{\mathrm{tau}}(g))$ yields
$$(\ell\Delta_{\Theta})(\mathbb{M}^{\Theta})/N =: \mu_N^{\text{ét}}(\mathbb{M}^{\Theta}) \xrightarrow{\sim} \mu_N^{\mathrm{Fr}}(\mathbb{M}^{\Theta}) := \ker\left((\ell\Delta_{\Theta}[\mu_N])(\mathbb{M}^{\Theta}) \twoheadrightarrow (\ell\Delta_{\Theta})(\mathbb{M}^{\Theta})\right)$$
- (**Discrete rigidity**) Any projective system
$$\cdots \to \mathbb{M}^{\Theta}_{N'} \xrightarrow{\gamma_{N',N}} \mathbb{M}^{\Theta}_{N} \to \cdots$$
is isomorphic to the projective system $\cdots \to \mathbb{M}^{\Theta}_{N'}(\underline{\underline{Y}}) \xrightarrow{\gamma_{N',N}} \mathbb{M}^{\Theta}_{N}(\underline{\underline{Y}}) \to \cdots$.
- (**Compatibility between CRI and labels on cusps**) The action on cusp labels
$$\Pi_C(\mathbb{M}^{\Theta})/\Pi_{\underline{X}}(\mathbb{M}^{\Theta}) \simeq \Delta_C(\mathbb{M}^{\Theta})/\Delta_{\underline{X}}(\mathbb{M}^{\Theta}) \simeq \mathbb{F}_{\ell}^{\rtimes\pm}$$
is compatible with CRI: $\mu_N^{\text{ét}}(\mathbb{M}^{\Theta}) \xrightarrow{\sim} \mu_N^{\mathrm{Fr}}(\mathbb{M}^{\Theta})$ $(\forall N \in \mathbb{Z}_{\geq 1})$.

- For $A \in \mathrm{Ob}(\mathcal{F}_0)$ and $f \in \mathcal{O}^\times(A^{\mathrm{birat}})$, assume f admits an N-th root on some tempered cover of X, and look at a pair of linear base-isomorphisms (called a fraction-pair)

$$s^\bullet, s_\bullet : A \to B, \quad \text{s.t.} \quad s^\bullet \cdot s_\bullet^{-1} = f$$

and their divisors have disjoint supports. Assume A is Frobenius-trivial (i.e. $\underline{A}$ equipped with the trivial line bundle). There are commutative diagrams in $\mathcal{F}_0$:

$$\begin{tikzcd} A_N \arrow[r, "s_N^\bullet"] \arrow[d] & B_N \arrow[d] \\ A \arrow[r, "s^\bullet"] & B \end{tikzcd} \qquad \begin{tikzcd} A_N \arrow[r, "(s_N)_\bullet"] \arrow[d] & B_N \arrow[d] \\ A \arrow[r, "s_\bullet"] & B \end{tikzcd}$$

with $(s_N^\bullet, (s_N)_\bullet)$ a fraction-pair for a function $f_N \in \mathcal{O}^\times(A_N^{\mathrm{birat}})$ so that $(f_N)^N = f|_{A_N}$.
- Assume that $\underline{A} \in \mathrm{Ob}(\mathcal{D}_X)$ corresponds to an open normal subgroup H of Π_X^{tp},

$$H_{\underline{A}} := \left(\Pi_X^{\mathrm{tp}} \twoheadrightarrow \mathrm{Aut}_{\mathcal{D}_X}(\underline{A})\right)(H),$$

$$H_A := \left(\mathrm{Aut}_{\mathcal{D}_X}(A)/\mathcal{O}^\times(A) \hookrightarrow \mathrm{Aut}_{\mathcal{D}_X}(\underline{A})\right)^{-1}(H_{\underline{A}}).$$

Similarly we define H_{A_N}, H_{B_N}.
- $\exists$ two actions $H_{B_N} \curvearrowright B_N$:

$$(s_N^\bullet)^{\mathrm{gp}}, \quad (s_N)_\bullet^{\mathrm{gp}} : H_{B_N} \to \mathrm{Aut}_{\mathcal{F}_0}(B_N),$$

such that $s_N^\bullet$ (resp. $(s_N)_\bullet$) is H_{B_N}-equivariant for $(s_N^\bullet)^{\mathrm{gp}}$ (resp. $(s_N)_\bullet^{\mathrm{gp}}$). The difference $\frac{(s_N^\bullet)^{\mathrm{gp}}}{(s_N)_\bullet^{\mathrm{gp}}}$ then determines a Kummer class

$$\eta^f \in H^1(H_{B_N}, \mu_N(B_N)).$$

- Suppose X is of type $(1,1)$. Recall the tempered Frobenioid

$$\underline{\underline{\mathcal{F}}} \to \mathcal{D} = \mathcal{B}^{\mathrm{tp}}(\underline{\underline{X}}).$$

- For theta functions, take

$$A = (\underline{\underline{\ddot{Y}}}, \mathcal{O}), \quad (s_N^{\bullet}, (s_N)_{\bullet}) = (s_{\ell N}, \tau_{\ell N}).$$

$\rightsquigarrow$ (modulo N) Kummer class $\underline{\underline{\ddot{\eta}}}^{\Theta}$ of an ℓ-th root of theta function.

- The Kummer class of theta function $\rightsquigarrow$ **cyclotomic rigidity isomorphism**:

$$(\ell \Delta_{\Theta})_S \otimes \mathbb{Z}/N \simeq \mu_N(S), \quad \forall S \in \mathrm{Ob}(\underline{\underline{\mathcal{F}}}).$$

- One can check that $(s_N^{\bullet})^{\mathrm{gp}}, (s_N)_{\bullet}^{\mathrm{gp}}$ factors through

$$\mathbb{E}_N := (s_N^{\bullet})^{\mathrm{gp}} \left(\mathrm{Im}(\Pi^{\mathrm{tp}}_{\underline{\underline{Y}}} \hookrightarrow \Pi^{\mathrm{tp}}_{\underline{\underline{X}}} \twoheadrightarrow \mathrm{Aut}_{\mathcal{D}}(\underline{B}_N)) \right) \cdot \mu_N(B) \subset \mathrm{Aut}_{\underline{\underline{\mathcal{F}}}}(B_N).$$

- Natural outer action

$$\ell \mathbb{Z} \simeq \Pi^{\mathrm{tp}}_{\underline{\underline{X}}} / \Pi^{\mathrm{tp}}_{\underline{\underline{Y}}} \to \mathrm{Out}(\mathbb{E}_N)$$

given by conjugation via $\Pi^{\mathrm{tp}}_{\underline{\underline{X}}} \twoheadrightarrow \mathrm{Aut}_{\mathcal{D}}(\underline{B}_N) \xrightarrow{(s_N^{\bullet})^{\mathrm{gp}}} \mathrm{Aut}_{\underline{\underline{\mathcal{F}}}}(B_N)$.

- Natural isomorphism of topological groups

$$\mathbb{E}_N \times_{\mathrm{Im}(\Pi^{\mathrm{tp}}_{\underline{\underline{Y}}})} \Pi^{\mathrm{tp}}_{\underline{\underline{Y}}} =: \mathbb{E}_N^{\Pi} \simeq \Pi^{\mathrm{tp}}_{\underline{\underline{Y}}}[\mu_N].$$

- $\mathbb{E}_N^{\Pi}$, the subgroup of $\mathrm{Out}(\mathbb{E}_N^{\Pi})$ generated by $\ell\mathbb{Z}$ and $k^{\times}$, the μ_N-conjugacy classes of $(s_N)_{\bullet}(\Pi^{\mathrm{tp}}_{\underline{\underline{\ddot{Y}}}})$, and varying $N \rightsquigarrow$ mono-theta environment $\mathbb{M}_{\Theta}(\underline{\underline{\mathcal{F}}})$.
- In particular, the previous rigidities hold for $\mathbb{M}_{\Theta}(\underline{\underline{\mathcal{F}}})$.

- Take an inversion $\iota_{\underline{X}}$ of $\underline{X}$ and let $\iota_{\underline{\underline{\ddot{X}}}}$ be the unique inversion of $\underline{\underline{X}}$ over k above $\iota_{\underline{X}}$.
  Then $\iota_{\underline{\underline{X}}}$ lifts to an inversion $\iota_{\underline{\underline{\ddot{Y}}}}$ of $\underline{\underline{\ddot{Y}}}$.
  The res. to decom. gp. $D_{\mu_-} \subset \Pi^{\mathrm{tp}}_{\underline{\underline{\ddot{Y}}}}$ of $(\mu_-)_{\underline{\underline{\ddot{Y}}}}$ of étale theta function lies in $\mu_{2\ell}$.
- Let $\Pi \simeq \Pi^{\mathrm{tp}}_{\underline{\underline{X}}}$ be a top. group. The μ_ℓ-multiples of the *reciprocal* of $\underline{\underline{\ddot{\eta}}}^{\Theta,\ell\mathbb{Z}\times\mu_2}(\Pi)$:

$$\underline{\underline{\theta}}(\Pi) \subset H^1(\Pi^{\mathrm{tp}}_{\underline{\underline{\ddot{Y}}}}(\Pi), \mu^{\text{ét}}(\Pi)).$$

  Elements whose certain positive integral power (up to torsion) lies in $\underline{\underline{\theta}}(\Pi)$:

$${}_\infty\underline{\underline{\theta}}(\Pi) \subset \varinjlim_{H} H^1(\Pi^{\mathrm{tp}}_{\underline{\underline{\ddot{Y}}}}(\Pi)|_H, \mu^{\text{ét}}(\Pi)).$$

- Let $(\iota, D) \simeq (\iota_{\underline{\underline{\ddot{Y}}}}, D_{\mu_-})$ be isomorph via Π. Restriction to D yields splittings

$$(\mathcal{O}^\times \cdot {}_\infty\underline{\underline{\theta}}^\iota(\Pi))/\mathcal{O}^\mu \simeq \mathcal{O}^{\times\mu}(\Pi) \times ({}_\infty\underline{\underline{\theta}}^\iota(\Pi)/\mathcal{O}^\mu)$$

  Mono-theta analogue:

$$(\mathcal{O}^\times \cdot {}_\infty\underline{\underline{\theta}}^\iota_{\mathrm{env}}(\mathbb{M}^\Theta(\Pi)))/\mathcal{O}^\mu \simeq \mathcal{O}^{\times\mu}(\mathbb{M}^\Theta(\Pi)) \times ({}_\infty\underline{\underline{\theta}}^\iota_{\mathrm{env}}(\mathbb{M}^\Theta(\Pi))/\mathcal{O}^\mu).$$

- $\Gamma^{\blacktriangleright}_{\underline{X}} \subset \Gamma_{\underline{X}}$ the unique connected subgraph which is a tree, stabilized by $\iota_{\underline{X}}$, and contains all vertices of $\Gamma_{\underline{X}}$.
  $\Gamma^{\bullet}_{\underline{X}} \leftrightarrow$ exactly one vertex of $\Gamma_{\underline{X}}$.
- Decomposition groups associated to subgraphs: $\Pi_{\bullet} \subset \Pi_{\blacktriangleright} \subset \Pi^{\mathrm{tp}}_{\underline{\underline{X}}} =: \Pi$.
- Look at $t \in \mathrm{LabCusp}^{\pm}(\Pi)$ and $\Pi_{\bullet t} \subset \Pi_{\blacktriangleright}$. For $\square = \bullet t, \blacktriangleright$, write
  $\Pi_{\ddot{\square}} = \Pi_{\square} \cap \Pi^{\mathrm{tp}}_{\underline{\underline{\ddot{Y}}}}, \quad \Delta_{\ddot{\square}} = \Delta \cap \Pi_{\ddot{\square}}$.
- Given a projective system $\mathbb{M}^{\Theta}$, have subgroups $\Pi_{\ddot{\blacktriangleright}}(\mathbb{M}^{\Theta}) \subset \Pi_{\blacktriangleright}(\mathbb{M}^{\Theta}) \subset \Pi(\mathbb{M}^{\Theta})$ corresponding to $\Pi_{\ddot{\blacktriangleright}} \subset \Pi_{\blacktriangleright} \subset \Pi$.

> Let X a hyperbolic curve over a p-adic local field k which admits stable reduction over $\mathcal{O}_k$. A decomposition/inertia group of Π_X is contained in Π^{tp}_X iff it is a decomposition/inertia group in Π^{tp}_X.

- Let I_t be a cuspidal ineria of $\Delta_{\bullet t}$. Suppose $\gamma' \in \Delta_{\underline{X}, \bullet t}$, $\gamma \in \widehat{\Delta}_{\underline{X}}$ and set $\delta = \gamma\gamma' \in \widehat{\Delta}_{\underline{X}}$. Well-defined decomposition group of μ_--translation of the cusp giving rise to I^{δ}_t:

$$D^{\delta}_{t,\mu_-} \subset \Pi^{\delta}_{\ddot{\blacktriangleright}} = \Pi^{\gamma}_{\ddot{\blacktriangleright}},$$

  compatible with the conjugacy of $\Delta_C/\Delta_{\underline{X}} \simeq \mathbb{F}_{\ell}^{\rtimes\pm}$.
- Get rid of dependence on $\iota_{\underline{\underline{\ddot{Y}}}} = \iota$: The restriction to D^{δ}_{t,μ_-} yields $\mu_{2\ell}, \mu$-orbits of elements

$$(\underline{\underline{\theta}}_{\mathrm{env}})_{|t|}((\mathbb{M}^{\Theta}_{\ddot{\blacktriangleright}})^{\gamma}) \subset {}_{\infty}(\underline{\underline{\theta}}_{\mathrm{env}})_{|t|}((\mathbb{M}^{\Theta}_{\ddot{\blacktriangleright}})^{\gamma}) \hookrightarrow \varinjlim_{H} H^1(G((\mathbb{M}^{\Theta}_{\ddot{\blacktriangleright}})^{\gamma})|_H, \mu^{\mathrm{Fr}}((\mathbb{M}^{\Theta}_{\ddot{\blacktriangleright}})^{\gamma})).$$

- (**Canonical splittings**) The restriction to D^{δ}_{0,μ_-} induces splittings

$$(\mathcal{O}^{\times} \cdot {}_{\infty}\underline{\underline{\theta}}^{\iota}_{\mathrm{env}}((\mathbb{M}^{\Theta}_{\ddot{\blacktriangleright}})^{\gamma}))/\mu \longrightarrow \mathcal{O}^{\times\mu}((\mathbb{M}^{\Theta}_{\ddot{\blacktriangleright}})^{\gamma}) \times ({}_{\infty}\underline{\underline{\theta}}^{\iota}_{\mathrm{env}}((\mathbb{M}^{\Theta}_{\ddot{\blacktriangleright}})^{\gamma})/\mu).$$

- (Constant Monoids $\simeq \mathcal{O}^{\triangleright}_{F_{\underline{v}}}$) $\mathbb{M}^{\Theta} := \mathbb{M}^{\Theta}(\underline{\underline{\mathcal{F}}}_{\underline{v}})$.

$$\begin{aligned}
\Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta})_{\underline{v}} &= \mathcal{O}^{\triangleright}(G_{\underline{v}}(\mathbb{M}^{\Theta})) \subset \varinjlim_{H} H^1(\Pi_{\underline{\underline{\ddot{Y}}}}(\mathbb{M}^{\Theta})|_H, \mu^{\mathrm{Fr}}(\mathbb{M}^{\Theta})), \\
\Psi_{\mathrm{cns}}(\mathcal{D}_{\underline{v}}) &= \mathcal{O}^{\triangleright}(G_{\underline{v}}(\mathcal{D}^{\vdash}_{\underline{v}})), \\
\Psi_{\mathrm{cns}}(\mathcal{F}_{\underline{v}}) &= \mathcal{O}^{\triangleright}_{\mathcal{C}_{\underline{v}}}(A^{\Theta}_{\infty}), \quad A^{\Theta}_{\infty} \text{ a universal covering pro-object of } \mathcal{D}_{\underline{v}}.
\end{aligned}$$

- Kummer map

$$\kappa : \Psi_{\mathrm{cns}}(\mathcal{F}_{\underline{v}}) \xrightarrow{\sim} \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta}) \simeq \Psi_{\mathrm{cns}}(\mathcal{D}_{\underline{v}})$$

-

$$\begin{aligned}
\Psi_{\mathrm{env}}(\mathbb{M}^{\Theta})_{\underline{v}} &= \left\{\Psi^{\iota}_{\mathrm{env}}(\mathbb{M}^{\Theta})_{\underline{v}} := \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta})^{\times}_{\underline{v}} \cdot \underline{\underline{\theta}}^{\iota}_{\mathrm{env}}(\mathbb{M}^{\Theta})^{\mathbb{N}}\right\}_{\iota}, \\
\Psi_{\mathcal{F}_{\mathrm{env}},\underline{v}} &= \left\{\Psi_{\mathcal{F}^{\Theta}_{\underline{v}},\alpha} := \mathcal{O}^{\times}_{\mathcal{C}^{\Theta}_{\underline{v}}}(A^{\Theta}_{\infty}) \cdot (\underline{\underline{\Theta}}^{\alpha}_{\underline{v}})^{\mathbb{N}}|_{A^{\Theta}_{\infty}}\right\}_{\alpha\in\Pi_{\underline{v}}},
\end{aligned}$$

- Matching the image of α under $\Pi_{\underline{v}} \twoheadrightarrow \ell\mathbb{Z}$ and $\iota \in \ell\mathbb{Z}$:

$$\kappa : {}_{(\infty)}\Psi_{\mathcal{F}_{\mathrm{env}},\underline{v}} \xrightarrow{\sim} {}_{(\infty)}\Psi_{\mathrm{env}}(\mathbb{M}^{\Theta}).$$

- For an isomorph $(G \curvearrowright \mathcal{O}^{\times\mu}) \simeq (G_{\underline{v}} \curvearrowright \mathcal{O}^{\times\mu}_{\overline{K}_{\underline{v}}})$,

$$\mathrm{Ism}(G) \quad (\subset \mathrm{Aut}_G(\mathcal{O}^{\times\mu})):$$

  G-equiv. automorphisms preserving $\mathrm{Im}((\mathcal{O}^{\times})^H \to \mathcal{O}^{\times\mu})$ for any open subgroup $H \subset G$.
- An Ism-orbit of $\mathcal{O}^{\times\mu} \simeq \mathcal{O}^{\times\mu}_{\overline{K}_{\underline{v}}}$ is called a $\times\mu$-**Kummer structure** on $(G \curvearrowright \mathcal{O}^{\times\mu})$.
- (The analogue for the constant monoids $\Psi_{\mathrm{cns}}(-)$ fails.)

(Multiradiality of split theta monoids)
Consider the full-poly-isomorphism $\mathbb{M}^{\Theta}(\Pi_{\underline{v}}) \simeq \mathbb{M}^{\Theta}(\underline{\underline{\mathcal{F}}}_{\underline{v}})$. The construction in $\Pi_{\underline{v}}$ of

$$_{(\infty)}\Psi_{\mathrm{env}}(\mathbb{M}^{\Theta}(\Pi_{\underline{v}})) \simeq {}_{(\infty)}\Psi_{\mathcal{F}_{\mathrm{env}},\underline{v}}$$

is compatible with arbitrary automorphisms of the pair $G(\mathbb{M}^{\Theta}) \curvearrowright \Psi^{\times\mu}_{\mathcal{F}_{\mathrm{env}},\underline{v}}$ $(\simeq G_{\underline{v}} \curvearrowright \mathcal{O}^{\times\mu}_{\overline{F}_{\underline{v}}})$.

- More concretely, this is the data

$$(\Pi, G, \mu^{\text{ét}}(\Pi), \mathbb{M}^{\Theta}(\Pi)) \longrightarrow (G \curvearrowright \mathcal{O}^{\times\mu}(G)),$$

where $\mu^{\text{ét}}(\Pi) \to \mathcal{O}^{\times\mu}(G)$ is the zero map.

- Recall: the action of $(\Delta_{C_{\underline{v}}}/\Delta_{\underline{X}_{\underline{v}}})(\mathbb{M}^{\Theta}) \simeq \mathbb{F}_{\ell}^{\rtimes\pm}$ on $\Pi_{\underline{X}_{\underline{v}}}(\mathbb{M}^{\Theta})$ induces isomorphisms among
$$G_{\underline{v}}(\mathbb{M}^{\Theta}_{\blacktriangleright\!\!\!\!\!\!\!\!\!\ddot{}})_t \curvearrowright \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta})_t, \quad t \in \mathbb{F}_{\ell}.$$
The subscript $\triangle$ indicates the identification of 0 with the diagonal of $\mathbb{F}_{\ell}^{*}$.
- $$\Psi_{\mathrm{gau}}(\mathbb{M}^{\Theta})_{\underline{v}} \quad = \quad \left\{\Psi_{\xi}(\mathbb{M}^{\Theta})_{\underline{v}} := \Psi^{\times}_{\mathrm{cns}}(\mathbb{M}^{\Theta}) \cdot \xi^{\mathbb{N}}\right\}_{\xi\in\prod_{|t|\in\mathbb{F}_{\ell}^{*}}(\underline{\underline{\theta}}_{\mathrm{env}})_{|t|}(\mathbb{M}^{\Theta}_{\ddot{\blacktriangleright}})} .$$
- The restriction $\underline{\underline{\theta}}^{\iota}(\Pi_{\underline{v}\ddot{\blacktriangleright}}) \xrightarrow{\sim} \underline{\underline{\theta}}_{|t|}(\Pi_{\underline{v}\ddot{\blacktriangleright}})$ induces **evaluation isomorphism**
$$\Psi_{\mathrm{env}}(\mathbb{M}^{\Theta})_{\underline{v}} \xrightarrow{\sim} \Psi_{\mathrm{gau}}(\mathbb{M}^{\Theta})_{\underline{v}}.$$
Intuitively, this is (modulo $\mu_{2\ell}$) $\underline{\underline{\Theta}}_{\underline{v}} \mapsto \{\underline{\underline{q}}_{\underline{v}}^{j^2}\}_{j\in\mathbb{F}_{\ell}^{*}}$.
- $\Psi_{\mathcal{F}_{\mathrm{gau}}} := \left\{\Psi_{\mathcal{F}_{\xi}}(\underline{\underline{\mathcal{F}}}_{\underline{v}}) = \mathrm{Im}\left(\Psi_{\xi}(\mathbb{M}^{\Theta})_{\underline{v}} \hookrightarrow \prod_{|t|\in\mathbb{F}_{\ell}^{*}} \Psi_{\mathrm{cns}}(\mathbb{M}^{\Theta})_{|t|} \xrightarrow{\sim} \prod_{|t|} \Psi_{\mathrm{cns}}(\mathcal{F}_{\underline{v}})\right)\right\}_{\xi}$.
- Global realified prime-strips (with Frobenioids $\mathcal{C}^{\Vdash}_{\mathrm{env}} \xrightarrow{\sim} \mathcal{C}^{\Vdash}_{\mathrm{gau}} \simeq \mathcal{C}^{\Vdash}_{\mathrm{mod}}$):
$$\mathfrak{F}^{\Vdash}_{\mathrm{env}} = \left(\mathcal{C}^{\Vdash}_{\mathrm{env}}, \mathrm{Prime}(\mathcal{C}^{\Vdash}_{\mathrm{env}}) \xrightarrow{\sim} \underline{\mathbb{V}}, \mathfrak{F}^{\vdash}_{\mathrm{env}} := \mathfrak{F}^{\vdash}(\Psi_{\mathcal{F}_{\mathrm{env}}}), \{\rho_{\underline{v}}\}\right), \quad \mathfrak{F}^{\Vdash}_{\mathrm{gau}} = \cdots .$$
- $\mathfrak{F}^{\vdash} = \{\mathcal{F}^{\vdash}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}} \rightsquigarrow$ split Frobenioids/prime-strips equipped with $\times\mu$-Kummer structures:
$$\mathfrak{F}^{\vdash\blacktriangleright\times\mu} = \{\mathcal{F}^{\vdash\blacktriangleright\times\mu}_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}, \quad \mathfrak{F}^{\Vdash\blacktriangleright\times\mu}_{\mathrm{env}}, \quad \mathfrak{F}^{\Vdash\blacktriangleright\times\mu}_{\mathrm{gau}}.$$
- Recall the localization functors $\mathcal{F}^{\circledast}_{\mathrm{mod}} \to \mathfrak{F}$ and thus (via $\mathcal{C}^{\Vdash}(\mathfrak{F}) \xrightarrow{\sim} \mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{mod}}$) the inclusion (with additive symmetry on the left and multiplicative symmetry on the right):
$$\boxed{\mathcal{C}^{\Vdash}_{\mathrm{gau}} \hookrightarrow \prod_{j\in\mathbb{F}_{\ell}^{*}} \mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{mod},j}.}$$

Log-links concern the **multiplication action** of theta values on the **additive module**:

$$\{\underline{\underline{q}}_{\underline{v}}^{j^2}\}_{j=1,\cdots,\frac{\ell-1}{2}} \curvearrowright \frac{1}{p_{\underline{v}}} \log(\mathcal{O}^{\times}_{F_{\underline{v}}}).$$

- Recall prime-strips:

$$\mathfrak{F} = \{\mathcal{F}_{\underline{v}} := \mathcal{C}_{\underline{v}}\}, \quad \mathfrak{D} = \{\mathcal{D}_{\underline{v}} \simeq \mathcal{B}^{\mathrm{tp}}(\Pi_{\underline{\underline{X}}_{\underline{v}}})\}, \quad \mathfrak{D}^{\vdash} = \{\mathcal{D}^{\vdash}_{\underline{v}} \simeq \mathcal{B}(G_{K_{\underline{v}}})\}.$$

- Consider the (**Galois equivariant**) log-operation (with codomain group-theoretic in the unit group):

$$\Psi_{\mathrm{cns}}(\mathcal{F}_{\underline{v}})^{\times} \xrightarrow{\mathfrak{log}} (\Psi_{\mathrm{cns}}(\mathcal{F}_{\underline{v}})^{\times})^{\mathrm{pf}} =: \underline{\mathfrak{log}}(\mathcal{F}_{\underline{v}}) \quad (\simeq \overline{F}_{\underline{v}}).$$

  $\rightsquigarrow$ monoid/Frobenioid:

$$\Psi_{\mathfrak{log}(\mathcal{F}_{\underline{v}})} \quad (\simeq \mathcal{O}^{\rhd}_{\overline{F}_{\underline{v}}}), \quad \mathfrak{log}(\mathcal{F}_{\underline{v}}) \quad (\simeq \mathcal{F}_{\underline{v}}).$$

- (i) A log-link on $\mathcal{F}$-prime-strips is defined to be a poly-isomorphism $\mathfrak{log}({}^{\ddagger}\mathfrak{F}) \xrightarrow{\sim} {}^{\dagger}\mathfrak{F}$:
  (ii) A **full log-link** between Hodge theaters is the collection of log-links on $\mathcal{F}$-prime-strips $\mathfrak{F}_{>}, \mathfrak{F}_{\succ}, \mathfrak{F}_{J}, \mathfrak{F}_{T}$ which lifts all isomorphisms on the bases:

$$\mathfrak{log} : {}^{0}\mathcal{HT} \to {}^{1}\mathcal{HT}.$$

- A log-link (with $\mathcal{O} = \mathcal{O}_{\overline{F}_{\underline{v}}}$) can be regarded as a transition map between ${}^{0}\mathcal{O}$ and ${}^{1}\mathcal{O}$.

$$\phi_{01} : {}^{0}\mathcal{O} \supset {}^{0}\mathcal{O}^{\times} \twoheadrightarrow ({}^{0}\mathcal{O}^{\times})^{\mathrm{pf}} \simeq {}^{0}\overline{F}_{\underline{v}} \xrightarrow{\text{full iso.}} {}^{1}\overline{F}_{\underline{v}} \supset {}^{1}\mathcal{O}.$$

  (The full poly-isomorphism is given by automorphisms of Π.)

- The Kummer isomorphisms $\kappa : \Psi_{\mathrm{cns}}(\mathfrak{F}) \xrightarrow{\sim} \Psi_{\mathrm{cns}}(\mathfrak{D})$ are **incompatible** with log-links.

- An object of $\mathcal{F}^{\circledast}_{\mathfrak{mod}}$ is of the form $\mathcal{J} = \{J_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}$ with $J_{\underline{v}} \subset K_{\underline{v}}$ a fractional ideal at $\underline{v}$ in $\underline{\mathbb{V}}$, almost all of which are the integer rings. Obviously, $\mathcal{F}^{\circledast}_{\mathfrak{mod}} \simeq \mathcal{F}^{\circledast}_{\mathrm{mod}}$.
- $F^{\times}_{\mathrm{mod}}$-torsor and a trivialization at each $\underline{v} \rightsquigarrow$ Frobenioid

$$\mathcal{F}^{\circledast}_{\mathrm{MOD}} \quad (\simeq \mathcal{F}^{\circledast}_{\mathfrak{mod}}).$$

- In the proof of the final inequality, we start with (the realification of) $\mathcal{F}^{\circledast}_{\mathrm{MOD}}$ (as part of the domain of Θ-link), and have part of the output (i.e. the mutiradial representation) in the product of copies of $\mathcal{F}^{\circledast}_{\mathfrak{mod}}$.
- $$\underline{\mathfrak{log}}(\mathcal{F}_{v_{\mathbb{Q}}}) = \oplus_{\underline{v} | v_{\mathbb{Q}}} \underline{\mathfrak{log}}(\mathcal{F}_{\underline{v}}), \quad \underline{\mathfrak{log}}(\mathcal{F}_{\mathbb{V}_{\mathbb{Q}}}) = \prod_{v_{\mathbb{Q}} \in \mathbb{V}_{\mathbb{Q}}} \underline{\mathfrak{log}}(\mathcal{F}_{v_{\mathbb{Q}}}).$$

  We embed the number field F_{mod} into it.
- To construct the additive version $\mathcal{F}^{\circledast}_{\mathfrak{mod}}$, we need the number field F_{mod} AND the additive modules $\Psi_{\mathfrak{log}(\mathcal{F}_{\underline{v}})} \bigcup \{0\} \simeq \mathcal{O}^{\triangleright}_{\overline{F}_{\underline{v}}}$.
- A nonzero element in the number field which is integral at all local places has to be a root of unity (and that log sends roots of unity to $0 \in F_{\mathrm{mod}}$) $\rightsquigarrow$

(Log-Kummer compatibility for number field)
The number field F_{mod} (as an entire set) is invariant under the log-links.

-
$$\begin{aligned}
\mathcal{I}(\mathcal{F}_{\underline{v}}) &= \frac{1}{2p_{\underline{v}}}\underline{\mathfrak{log}}\left((\Psi_{\mathrm{cns}}(\mathcal{F}_{\underline{v}})^{\times})^{G_{\underline{v}}}\right) \subset \underline{\mathfrak{log}}(\mathcal{F}_{\underline{v}}),\\
\mathcal{I}(\mathcal{D}_{\underline{v}}) &= \mathcal{I}(\mathcal{F}_{\underline{v}}(\mathfrak{D})),\\
\mathcal{I}(\mathcal{F}_{\underline{v}}^{\vdash\times\mu}) &= \cdots,\\
\mathcal{I}(\mathcal{D}_{\underline{v}}^{\vdash}) &= \mathcal{I}(G_{\underline{v}}) = \frac{1}{2p_{\underline{v}}}\underline{\mathfrak{log}}(\mathcal{O}_{K_{\underline{v}}}^{\times}(G_{\underline{v}})).
\end{aligned}$$

- For $(*) = \mathcal{F}, \mathcal{F}^{\vdash\times\mu}, \mathcal{D}, \mathcal{D}^{\vdash}$, define the (tensor-packets of) log-shells.

$$\begin{aligned}
\mathcal{I}({}^{i}(*)_{v_{\mathbb{Q}}}) &= \oplus_{\underline{v}|v_{\mathbb{Q}}}\mathcal{I}({}^{i}(*)_{\underline{v}}), \quad \forall i \in I,\\
\mathcal{I}({}^{I}(*)_{v_{\mathbb{Q}}}) &= \otimes_{i\in I}\mathcal{I}({}^{i}(*)_{v_{\mathbb{Q}}}),\\
\mathcal{I}({}^{I,j}(*)_{\underline{v}}) &= \mathcal{I}({}^{j}(*)_{\underline{v}}) \otimes \mathcal{I}({}^{I\setminus j}(*)_{v_{\mathbb{Q}}}),\\
\mathcal{I}^{\mathbb{Q}}({}^{(-)}(*)_{\mathbb{V}_{\mathbb{Q}}}) &= \prod_{v_{\mathbb{Q}}\in\mathbb{V}_{\mathbb{Q}}} \mathcal{I}^{\mathbb{Q}}({}^{(-)}(*)_{v_{\mathbb{Q}}}).
\end{aligned}$$

- On an arbitrary compact open subset of a finite extension of $\mathbb{Q}_p$, there is a well-defined volume $\mu(-)$. The log-volume (normalized so that multiplication by p induces $-\log(p)$):

$$\mu^{\log}(-) := \log(\mu(-)).$$

  For an object $\mathcal{J} = \{J_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}}$ of $\mathcal{F}_{\mathfrak{mod}}^{\circledast}$, regarded as an element in $\mathcal{I}^{\mathbb{Q}}({}^{I}\mathcal{F}_{\mathbb{V}_{\mathbb{Q}}})$, we take the sum $\mu_{I,\mathbb{V}_{\mathbb{Q}}}^{\log}(\mathcal{J})$ of the log-volumes of the $J_{\underline{v}}$ as $\underline{v}$ varies.
- $\mu_{I,\mathbb{V}_{\mathbb{Q}}}^{\log}(\mathcal{J})$ is invariant by multiplication by non-zero elements of the number field, and is equal to the degree of the arithmetic line bundle corresponding to $\mathcal{J}$.
- The log-volumes on the log-shells are invariant under the log-links.

- A **procession** in a category is a diagram

$$P_1 \hookrightarrow P_2 \hookrightarrow \cdots \hookrightarrow P_n$$

with P_j a j-capsule of objects and $\hookrightarrow$ the collection of all capsule-full poly-morphisms.

- Given a full log-link $\mathfrak{log} : {}^{m-1}\mathcal{HT} \to {}^{m}\mathcal{HT}$,

$$^{m}\Psi_{\mathcal{F}_{\mathrm{LGP}},\underline{v}} := \mathrm{Im}\left(\Psi_{\mathcal{F}_{\mathrm{gau}}}({}^{m}\mathcal{F}_{\underline{v}}) \hookrightarrow \prod_{j\in\mathbb{F}_\ell^{*}} (\Psi_{{}^{m}\mathcal{F}_{\underline{v}}})_j \to \prod_{j\in\mathbb{F}_\ell^{*}} \underline{\mathfrak{log}}({}^{m-1}\mathcal{F}_{\underline{v}})_j\right),$$

$$^{m}\Psi^{\mathcal{D}}_{\mathcal{F}_{\mathrm{LGP}},\underline{v}} := \mathrm{Im}\left(\Psi_{\mathrm{env}}({}^{m}\mathcal{D}_{\underline{v}}) \hookrightarrow \prod_{j\in\mathbb{F}_\ell^{*}} \Psi_{\mathrm{cns}}({}^{m}\mathcal{D}_{\underline{v}})_j \to \prod_{j\in\mathbb{F}_\ell^{*}} \underline{\mathfrak{log}}({}^{m-1}\mathcal{F}_{\underline{v}}(\mathcal{D}_{\underline{v}}))_j\right),$$

so that they are compatible with the processions on the ${}^{m-1}\mathcal{F}_j$ and the ${}^{m}\mathcal{F}_j$.

- In more concrete terms, this is

$$\mathcal{I}({}^{\mathbb{S}_1}\mathcal{D}_{\underline{v}}) \hookrightarrow$$

$$\left(\underline{\underline{q}}_{\underline{v}}^{1} \curvearrowright \mathcal{I}({}^{\mathbb{S}_2,1}\mathcal{D}_{\underline{v}})\right) \hookrightarrow \left(\underline{\underline{q}}_{\underline{v}}^{2^2} \curvearrowright \mathcal{I}({}^{\mathbb{S}_3,2}\mathcal{D}_{\underline{v}})\right) \hookrightarrow \cdots \hookrightarrow \left(\underline{\underline{q}}_{\underline{v}}^{(\ell^{*})^2} \curvearrowright \mathcal{I}({}^{\mathbb{S}_{\ell^{*}+1},\ell^{*}}\mathcal{D}_{\underline{v}})\right).$$

(The first component $\mathcal{I}({}^{\mathbb{S}_1}\mathcal{D}_{\underline{v}}) = \mathcal{I}(\mathcal{D}_{\underline{v}})$ is considered as coric data.)

- Consider the splitting monoid $\Psi^{\perp}_{\mathcal{F}_{\mathrm{LGP}},\underline{v}}$ of the monoid $\Psi_{\mathcal{F}_{\mathrm{LGP}},\underline{v}}$ given by the split Frobenioid $\mathcal{F}^{\vdash}_{\underline{v}}$, i.e. the value group portion corresponding to $\{\underline{\underline{q}}_{\underline{v}}^{j^2}\}_{j\in\mathbb{F}_\ell^{*}}$.

- Fact $\Psi^{\perp}_{\mathcal{F}_{\mathrm{LGP}},\underline{v}} \cap \Psi^{\times}_{\mathcal{F}_{\mathrm{LGP}},\underline{v}} = \mu_{2\ell} \quad \rightsquigarrow$

$$\boxed{\begin{array}{c}\textbf{(Log-Kummer compatibility for theta values)}\\ \Psi^{\perp}_{\mathcal{F}_{\mathrm{LGP}},\underline{v}} \text{ (defined up to torsion, hence) is invariant under log-links.}\end{array}}$$

The coric data (top. gp. $\curvearrowright$ top. monoid)

$$G_{\underline{v}} \curvearrowright \mathcal{O}^{\times\mu}_{\overline{F}_{\underline{v}}}$$

is compatible with theta functions/values because of the cyclotomic rigidity isomorphisms.

- $$\mathcal{C}^{\Vdash}_{\mathrm{LGP}} = \mathrm{Im}(\mathcal{C}^{\Vdash}_{\mathrm{gau}}(\mathcal{HT}^{\Theta}) \hookrightarrow \prod_j \mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{mod},j} \xrightarrow{\sim} \prod_j \mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{MOD},j}).$$

$$\mathfrak{F}^{\Vdash}_{\mathrm{LGP}} = \left(\mathcal{C}^{\Vdash}_{\mathrm{LGP}}, \mathrm{Prime}({}^{\dagger}\mathcal{C}^{\Vdash}_{\mathrm{LGP}}) \xrightarrow{\sim} \underline{\mathbb{V}}, \mathfrak{F}^{\vdash}_{\mathrm{LGP}} := \mathfrak{F}^{\vdash}(\Psi_{\mathcal{F}_{\mathrm{LGP}}}), \{\rho_{\underline{v}}\}\right).$$

- A Θ-pilot object P_{Θ} is an element in the the product of Frobenioid associated to F_{mod}

$$P_{\Theta} \in \prod_{j \in \mathbb{F}_{\ell}^{*}} (\mathcal{F}^{\circledast}_{\mathrm{MOD}})_j$$

determined by any collection of generators of the monoids $\Psi^{\perp}_{\mathcal{F}_{\mathrm{LGP}},\underline{v}}$ for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$. Intuitively, this is, at each $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, the data

$$\left(\{\underline{\underline{q}}^{j^2}_{\underline{v}}\}_{j \in \mathbb{F}_{\ell}^{*}}\right)^{\mathbb{N}} \cdot \mathcal{O}^{\times\mu}_{\overline{F}_{\underline{v}}}.$$

- P_{Θ} gives rise to a generator of the global realified prime-strip:

$$P_{\Theta}^{\Vdash\blacktriangleright\times\mu} \in \mathfrak{F}^{\Vdash\blacktriangleright\times\mu}_{\mathrm{LGP}}.$$

The column ${}^{\bullet}\mathcal{HT}^{\mathcal{D}} = \{{}^{m}\mathcal{HT}^{\mathcal{D}}\}_{m\in\mathbb{Z}}$ linked by the full poly-isomorphism (induced by log-link). It carries processions. (The log-volume on a procession is normalized by taking elementary average.)

- ${}^{\bullet}\mathcal{HT}^{\mathcal{D}} \rightsquigarrow$ the following data ${}^{\bullet}\mathfrak{R}^{\mathrm{LGP}}$:
  - (a) The topological modules carrying the procession-normalized log-volume map
  $$\mathcal{I}({}^{\mathbb{S}_{j+1},j}({}^{\bullet}\mathcal{D}^{\vdash}_{\underline{v}})) \subset \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}_{j+1},j}({}^{\bullet}\mathcal{D}^{\vdash}_{\underline{v}})).$$
  - (b) The number field embedded into the product of local fields
  $${}^{\bullet}\overline{\mathbb{M}}^{\circledast\mathcal{D}}_{\mathfrak{mod},j} \subset \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}_{j+1}}({}^{\bullet}\mathcal{D}^{\vdash}_{\underline{\mathbb{V}}_{\mathbb{Q}}})), \quad j \in \mathbb{F}^{\circledast}_{\ell}.$$
  - (c) For each $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, the splitting monoid + action on the log-shells by multiplication:
  $${}^{\bullet}\Psi^{\perp\mathcal{D}}_{\mathcal{F}_{\mathrm{LGP}},\underline{v}} \subset \prod_{j\in\mathbb{F}^{\circledast}_{\ell}} \mathcal{I}({}^{\mathbb{S}_{j+1},j}({}^{\bullet}\mathcal{D}^{\vdash}_{\underline{v}})).$$

- (Ind2) is the actions of $\mathrm{Ism}(G_{\underline{v}})$ on each component of $\mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}_{j+1},j}({}^{\bullet}\mathcal{D}^{\vdash}_{\underline{v}}))$, for every $\underline{v}$.
  (Ind1) is the automorphisms $\mathrm{Aut}(\mathrm{Prc}({}^{\bullet}\mathfrak{D}^{\vdash}_{T}))$.
  (Ind3) refers to the following inclusion, called log-Kummer upper semi-compatibility:
  $$(\mathrm{Ind1,2} \curvearrowright) \quad \mathcal{I}({}^{\mathbb{S}_{j+1,j}}({}^{\bullet}\mathcal{D}^{\vdash}_{\underline{v}})) \supset \bigcup_{n\in\{0,1\},m\in\mathbb{Z}} \mathrm{Km}\circ\phi^{n}_{mm+1}({}^{m}\Psi_{\mathrm{cns}}(\mathcal{F}_{\underline{v}})^{G_{\underline{v}}}).$$

  (**RHS is invariant under log-links, hence carries obvious actions given by (Ind1,2)!**)

- (**Multiradial representation of theta values**) Let P_Θ be a Θ-pilot object.

$$U_\Theta := \bigcup_{\mathrm{Ind1,2}} \left(\bigcup_{\mathrm{Ind3}} \mathrm{Km}(P_\Theta) \right) \subset \prod_{j \in \mathbb{F}_\ell^{*}} \mathcal{I}^{\mathbb{Q}}(\mathbb{S}_{j+1}({}^\bullet\mathcal{D}_{\triangle,\mathbb{V}_{\mathbb{Q}}})),$$

i.e. the union of the translations by the actions of Aut$(\mathrm{Prc}({}^\bullet\mathfrak{D}^{\vdash}_{T}))$ and the Ism-action (for each $\underline{v} \in \underline{\mathbb{V}}$) on each direct summand of the $j+1$ factors of

$$\mathcal{I}^{\mathbb{Q}}(\mathbb{S}_{j+1,j}({}^\bullet\mathcal{D}_{\underline{v}})) \simeq \mathcal{I}^{\mathbb{Q}}(\mathbb{S}_{j+1,j}({}^\bullet\mathcal{D}^{\vdash}_{\underline{v}}))$$

of the set $\mathrm{Km}(P_\Theta)$, equipped with their actions on the tensor-packets of log-shells.

- More concretely, at a bad place $\underline{v}$,

$$(U_\Theta)_{\underline{v}} = \bigcup_{j \in \mathbb{F}_\ell^{*}} \bigcup_{n \in \{0,1\}, m \in \mathbb{Z}} \mathrm{Km} \circ \phi^{n}_{mm+1}(\underline{\underline{q}}_{\underline{v}}^{j^2} \cdot {}^{m}\mathcal{O}_{\underline{v}}).$$

> **Theorem (Multiradiality).**
> The **construction** of multiradial representation U_Θ of Θ-pilot objects is invariant under arbitrary automorphism of the abstract prime-strip $\mathfrak{F}^{\Vdash\blacktriangleright\times\mu}$.

- Test object: A q-pilot object is an element in the global realified Frobenioid

$$P_q \in \mathcal{C}^{\Vdash}_{\triangle}$$

determined by any collection of generators (up to torsion) of the splitting monoids of $\mathcal{F}^{\vdash}_{\triangle,\underline{v}}$ for $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$.

- P_q gives rise to a generator of the global realified prime-strip:

$$P_q^{\Vdash\blacktriangleright\times\mu} \in \mathfrak{F}_{\triangle}^{\Vdash\blacktriangleright\times\mu}.$$

- Θ-link=the **full** poly-isomorphism

$$\Theta_{\mathrm{LGP}} : {}^{\ddagger}\mathfrak{F}_{\mathrm{LGP}}^{\Vdash\blacktriangleright\times\mu} \xrightarrow{\sim} {}^{\dagger}\mathfrak{F}_{\triangle}^{\Vdash\blacktriangleright\times\mu}.$$

At $\underline{v} \in \underline{\mathbb{V}}^{\mathrm{bad}}$, this (intuitively) is the identification of $P_{\Theta}^{\Vdash\blacktriangleright\times\mu}$ with $P_q^{\Vdash\blacktriangleright\times\mu}$:

$$\left(\{\underline{\underline{q}}_{\underline{v}}^{j^2}\}_{j\in\mathbb{F}_\ell^{*}}\right)^{\mathbb{N}} \cdot \mathcal{O}_{\overline{F}_{\underline{v}}}^{\times\mu} \leftrightarrow \underline{\underline{q}}_{\underline{v}}^{\mathbb{N}} \cdot \mathcal{O}_{\overline{F}_{\underline{v}}}^{\times\mu}.$$

Caution: P_q is not equipped with local trivializations at each $\underline{v}$, while P_Θ is (because of the canonical splitting of the integer rings given by theta values).

At each $v_{\mathbb{Q}}$, define the **holomorphic hull** of $(U_\Theta)_{v_{\mathbb{Q}}}$ as the smallest subset of $\mathcal{I}^{\mathbb{Q}}({}^{(-)}\mathcal{F}_{v_{\mathbb{Q}}})$ containing $(U_\Theta)_{v_{\mathbb{Q}}}$, which is of the form

$$(\overline{U}_\Theta)_{v_{\mathbb{Q}}} = \lambda \cdot \mathcal{O}_{{}^{(-)}\mathcal{F}_{v_{\mathbb{Q}}}}$$

with $\lambda \in \mathcal{I}^{\mathbb{Q}}({}^{(-)}\mathcal{F}_{v_{\mathbb{Q}}})$, which is non-zero in each direct summand of the decomposition of $\mathcal{I}^{\mathbb{Q}}({}^{(-)}\mathcal{F}_{v_{\mathbb{Q}}})$ into direct sum of local fields.

Theorem. Let $P_q \in \mathcal{C}_{\triangle}^{\Vdash}$ be a q-pilot object. Let

$$\overline{U}_\Theta \subset \prod_{j \in \mathbb{F}_\ell^*} \mathcal{I}^{\mathbb{Q}}({}^{\mathbb{S}_{j+1}}({}^{\bullet}\mathcal{D}_{\triangle, \mathbb{V}_{\mathbb{Q}}}))$$

be the holomorphic hull of the multiradial representation U_Θ of a Θ-pilot object P_Θ. Then

$$\mu^{\log}(\underline{\underline{\Theta}}) > \mu^{\log}(\underline{\underline{q}}).$$

(1) First consider the 0-column of Hodge theaters ${}^{0,\bullet}\mathcal{HT}$.

$$ {}^{0}\mu({}^{0}U_{\Theta}) \geq {}^{0}\mu(\kappa({}^{0,0}P_{\Theta})) = {}^{0}\mu({}^{0,0}P_{\Theta}). $$

(2) Now regard the domain and codomain of $\Theta_{\mathrm{LGP}} : {}^{0,0}\mathfrak{F}_{\mathrm{LGP}}^{\Vdash\blacktriangleright\times\mu} \xrightarrow{\sim} {}^{1,0}\mathfrak{F}_{\triangle}^{\Vdash\blacktriangleright\times\mu}$ as associated to the Hodge theaters ${}^{0,0}\mathcal{HT}$ and ${}^{1,0}\mathcal{HT}$, respectively. (Keep in mind that the use of the Θ_{LGP}-link implicitly requires that we regard both sides as abstract prime-strips.) We then have the association of log-volumes

$$ \alpha : {}^{0}\mathbb{R}_{\geq 0} \xrightarrow{\sim} {}^{1}\mathbb{R}_{\geq 0}, \quad {}^{0}\mu({}^{0,0}P_{\Theta}) \mapsto {}^{1,0}\deg({}^{1,0}P_{q}) \quad (= \mu^{\log}(\underline{\underline{q}})). $$

(3) Let ${}^{1,0}P_{\Theta} \in {}^{1,0}\mathcal{C}_{\mathrm{LGP}}^{\Vdash}$ be a Θ-pilot object. Set

$$ {}^{1}U_{\Theta} = \bigcup_{\mathrm{Ind}1,\ 2,\ 3} \kappa({}^{1,0}P_{\Theta}). $$

(4) ${}^{1}\overline{U}_{\Theta}$ is by definition the holomorphic hull of ${}^{0}U_{\Theta}$, Then

$$ ({}^{1}\mu({}^{1}U_{\Theta}) <) \quad {}^{1}\mu({}^{1}\overline{U}_{\Theta}) = {}^{1,0}\deg({}^{1}\overline{U}_{\Theta}) \quad (= \mu^{\log}(\underline{\underline{\Theta}})). $$

(5) By Multiradiality Theorem, the construction of ${}^{0}U_{\Theta}$ is compatible with Θ_{LGP}, hence has no interference with the holomorphic structure of ${}^{1,0}\mathcal{HT}$. Then, the assignment

$$ \mu^{0}({}^{0}U_{\Theta}) \mapsto \mu^{1}({}^{1}\overline{U}_{\Theta}). $$

is compatible the map α. Assembling what's above, one gets

$$ {}^{1,0}\deg({}^{1}\overline{U}_{\Theta}) > {}^{1,0}\deg({}^{1,0}P_{q}), $$

i.e. $\mu^{\log}(\underline{\underline{\Theta}}) > \mu^{\log}(\underline{\underline{q}})$.

Let us summarize the proof by drawing a picture.
Write

$$ {}^{i,\bullet}\mathcal{HT} = \{ \cdots {}^{i,-1}\mathcal{HT} \overset{\mathfrak{log}}{\to} {}^{i,0}\mathcal{HT} \cdots \}. $$

> Caution: The labels on $\mathbb{R}$ below are for illustration purpose only.
> $\nexists$ **relations among these copies of $\mathbb{R}$ in general**.

$$
\begin{tikzcd}
{}^{1,0}\mathcal{HT} \arrow[r, hook] \arrow[d] & {}^{1,\bullet}\mathcal{HT} \arrow[dd, "{}^{1}\mu"] & {}^{0,\bullet}\mathcal{HT} \arrow[dd, "{}^{0}\mu"] & {}^{0,0}\mathcal{HT} \arrow[l, hook'] \arrow[d] & {}^{1,0}\mathcal{HT} \arrow[d] \\
{}^{1,0}\mathfrak{F}_{\triangle}^{\Vdash\blacktriangleright\times\mu} \arrow[d, "{}^{1,0}\mathrm{deg}"'] \arrow[rrrr, bend left, "\simeq"] & & & {}^{0,0}\mathfrak{F}_{\mathrm{LGP}}^{\Vdash\blacktriangleright\times\mu} \arrow[r, "\Theta_{\mathrm{LGP}}"] \arrow[d, "{}^{0}\mu"] & {}^{1,0}\mathfrak{F}_{\triangle}^{\Vdash\blacktriangleright\times\mu} \arrow[d, "{}^{1,0}\mathrm{deg}"] \\
{}^{1}\mathbb{R}_{\geq 0} \arrow[drr] & {}^{1}\mathbb{R}_{\geq 0} \arrow[l, "h"'] \arrow[dr] & {}^{0}\mathbb{R}_{\geq 0} \arrow[l, "\gamma"'] \arrow[d] & {}^{0}\mathbb{R}_{\geq 0} \arrow[l, "\kappa^{\log}"'] \arrow[r, "\alpha"] \arrow[dl] & {}^{1}\mathbb{R}_{\geq 0} \arrow[dll] \\
& & \mathbb{R}^{\vdash}_{\geq 0} & &
\end{tikzcd}
$$

Here the arrow $\kappa^{\log}$ represents the operation ${}^{0,0}P_{\Theta} \mapsto {}^{0}U_{\Theta}$, the arrow γ denotes the operation ${}^{0}U_{\Theta} \mapsto {}^{1}U_{\Theta}$, $\mathbb{R}^{\vdash}_{\geq 0}$ denotes the codomain of the log-volume map $\mu^{\vdash}$ on the mono-analytic log-shells, and the arrow $h : {}^{1}U_{\Theta} \mapsto {}^{1}\overline{U}_{\Theta}$ denotes the operation of taking holomorphic hull.